# ASYMPTOTIC RELATIVE EFFICIENCIES OF CERTAIN TEST CRITERIA BASED ON FIRST AND SECOND DIFFERENCES

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## 1. Introduction

In a recent paper Kamat and Sathe (1962) have discussed the asymptotic relative power of certain test criteria based on first and second successive differences. To state briefly, let $x_i \, (i = 1, 2, 3, \cdots, n)$ be a sample-sequence of $n$ observations. Then the following estimates of dispersion are defined as usual.

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2, \quad \bar{x} = \sum_{i=1}^{n} \frac{x_i}{n},$$

$$\delta^2 = \frac{1}{n-1} \sum_{i=1}^{n-1} (x_i - x_{i+1})^2, \quad d = \frac{1}{n-1} \sum_{i=1}^{n-1} | x_i - x_{i+1} |,$$

$$\delta_2^2 = \frac{1}{n-2} \sum_{i=1}^{n-2} (x_i - 2x_{i+1} + x_{i+2})^2,$$

$$d_2 = \frac{1}{n-2} \sum_{i=1}^{n-2} | x_i - 2x_{i+1} + x_{i+2} | \tag{1}$$

Making use of these measures of dispersion, Tintner (1940), Von Neumann (1941) and Kamat (1953, 1958) suggested the following ratio criteria for testing (the existence of) serial correlation or trend in the sequence.

$$w^2 = \frac{\delta^2}{s^2}, \qquad W = \frac{d}{s},$$

$$w_2^2 = \frac{\delta_2^2}{s^2}, \qquad W_2 = \frac{d_2}{s},$$

$$u^2 = \frac{\delta_2^2}{\delta^2}, \qquad U = \frac{d_2}{d}. \tag{2}$$

[Readers interested in the relevant literature are referred to Kamat (1958) where a complete list of references is given.]

Under the hypothesis of randomness, when $x_i$ are independent normal $(\mu, \sigma)$, these ratio statistics are all asymptotically normally distributed (*see e.g.*, Kamat, 1958). Let this hypothesis be denoted by

$$H_0 \qquad \rho(x_i, x_j) = 0, \quad i \neq j. \tag{3}$$

In the paper mentioned above (1962) we have shown that they are asymptotically normal also under the alternative of serial dependence between successive observations defined by

$$H_1 \qquad \rho(x_i, x_{i+1}) = \rho$$
$$\rho(x_i, x_j) = 0, \text{ otherwise}, \tag{4}$$

for all $i$ and $j$ when $x_i$ are jointly normally distributed. (In other words $\{x_i\}$ is a stationary normal time series.) In that paper we considered the asymptotic power of these criteria for the alternative $H_1$ by deriving the asymptotic means and variances, and also discussed the asymptotic relative efficiency by applying the Pitman procedure as modified by Noèther (1955). In this paper we discuss the asymptotic relative efficiency of these six criteria under other alternatives of serial correlation which are more general than $H_1$ and under certain alternatives of trend, by using this latter procedure.

## 2. Preliminary Remarks

Before we define and consider these alternative hypotheses it is necessary to make a few preliminary remarks. First, it will be assumed in the following that

$$\epsilon(x_i) = 0, \qquad \epsilon(x_i^2) = 1, \tag{5}$$

which involves no loss of generality. Secondly, we shall assume the asymptotic normality of the six ratio criteria under all the alternatives considered below. This is necessary for using the Noéther-Pitman method, and *it can be proved* by applying the theorems given in Kamat and Sathe (1962) or Kamat (1958) as the case may be. Thirdly, we state briefly the Noéther (1955) procedure. Suppose a null hypothesis, $\theta = \theta_0$, is to be tested against the alternative, $\theta = \theta_n > \theta_0$, where $\theta_n \to \theta_0$ as $n \to \infty$.

Let $T$ be a test statistic which is asymptotically normally distributed under both null and alternative hypotheses, and let $\epsilon(T) = \psi(\theta)$ and $\text{Var}(T) = \sigma^2(\theta)$. Further let $m$ be the least integer for which $\psi^{(m)}(\theta_0) = (\partial^m \psi(\theta)/\partial\theta^m)_{\theta=\theta_0} \neq 0$, and let $\delta\,(> 0)$ be defined such that

$$\lim_{n\to\infty} \frac{n^{-m\delta}\,\psi^{(m)}(\theta_0)}{\sigma(\theta_0)} = c. \tag{6}$$

Then Noéther considers the power of the test based on $T$ with respect to the alternative $\theta_n = \theta_0 + kn^{-\delta}$, where $k$ is an arbitrary positive constant. In most cases (as, for instance, for the six ratio criteria defined above), $m\delta = \frac{1}{2}$. In that case, if $T_1$ and $T_2$ are the test statistics to be compared, then $E_{21}$, the asymptotic relative efficiency of $T_2$ with respect to $T_1$ is given by

$$E_{21} = \left(\frac{c_2}{c_1}\right)^2 \tag{7}$$

provided $m_1 = m_2$ and $\delta_1 = \delta_2$. Here $(m_1, \delta_1, c_1)$ and $(m_2, \delta_2, c_2)$ are the constants defined by (or implied in) the relation (6), for $T_1$ and $T_2$ respectively.

Finally, we give below the asymptotic variances of the six ratio criteria under the null hypothesis $H_0$; these are required for applying the Noéther procedure.

$$\begin{aligned}
\text{Var}\,(w^2) &= 4n^{-1}\\
\text{Var}\,(W) &= 0{\cdot}4156\,n^{-1} + 0\,(n^{-2})\\
\text{Var}\,(w_2{}^2) &= 68n^{-1} + 0\,(n^{-2})\\
\text{Var}\,(W_2) &= 2{\cdot}1479n^{-1} + 0\,(n^{-2})\\
\text{Var}\,(u^2) &= 2n^{-1} + 0\,(n^{-2})\\
\text{Var}\,(U) &= 0{\cdot}2799\,n^{-1} + 0\,(n^{-2})
\end{aligned} \tag{8}$$

## 3. Alternatives of Serial Correlation

Two alternative hypotheses of serial dependence which are generalisations of the alternative $H_1$ defined in (4) above, will be considered. First, we take up the hypothesis

$$H_2 \qquad \begin{aligned} \rho\,(x_i, x_{i+j}) &= \rho, && j = 1, 2, \ldots, k\,(>1)\\ &= 0, && \text{otherwise.}\end{aligned} \tag{9}$$

It can be easily shown that

$$\epsilon\,(s^2) = 1 - \frac{(2nk - k^2 - k)}{n\,(n-1)} = 1 + 0\,(n^{-1}), \tag{10}$$

for large $n$, and for a fixed $k$. The following results are derived immediately.

$$\epsilon(s) = 1 + 0(n^{-1}),$$

$$\epsilon(\delta^2) = 2(1-\rho), \qquad \epsilon(\delta_2^2) = 6(1-\rho),$$

$$\epsilon(d) = \frac{2}{\sqrt{\pi}}(1-\rho)^{\frac{1}{2}}, \qquad \epsilon(d_2) = \sqrt{\frac{12}{\pi}}(1-\rho)^{\frac{1}{2}}.$$

Hence the expectations of the six ratio statistics under $H_2$, for large $n$, can be written down as follows:—

$$\epsilon(w^2) \cong 2(1-\rho), \qquad \epsilon(W) \cong \frac{2}{\sqrt{\pi}}(1-\rho)^{\frac{1}{2}},$$

$$\epsilon(w_2^2) \cong 6(1-\rho), \qquad \epsilon(W_2) \cong \sqrt{\frac{12}{\pi}}(1-\rho)^{\frac{1}{2}},$$

$$\epsilon(u^2) \cong 3 \qquad \epsilon(U) \cong \sqrt{3}. \tag{11}$$

From the last two results it is clear that the statistics $u^2$ and $U$ are not useful for discriminating between $H_2$ and $H_0$. The Noéther procedure is therefore applied to the first four criteria. From (8) and (11) we find that for all of them $m = 1$ and $\delta = \frac{1}{2}$. Further we obtain for them the following values of $c^{-2}$.

$$w^2 : 1 \qquad W : 1\cdot3058$$

$$w_2^2 : 17/9 \qquad W_2 : 2\cdot2500 \tag{12}$$

The asymptotic relative efficiencies of the four criteria (relative to $w^2$ are therefore given by

$$w^2 : 1\cdot000 \qquad W : 0\cdot766$$

$$w_2^2 : 0\cdot529 \qquad W_2 : 0\cdot444 \tag{13}$$

It is interesting to compare these relative efficiencies for the alternative $H_2$ with those given below which were obtained by us (1962) for the alternative $H_1$.

$$w^2 : 1\cdot000 \qquad W : 0\cdot766$$

$$w_2^2 : 0\cdot941 \qquad W_2 : 0\cdot790$$

$$u^2 : 0\cdot500 \qquad U : 0\cdot298 \tag{14}$$

While the asymptotic relative efficiency of $W$ is the same for $H_1$ and $H_2$, that of $w_2^2$ and $W_2$ is much lower for $H_2$ than for $H_1$; and $u^2$ and $U$ have zero asymptotic efficiency of detecting $H_2$.

The second alternative hypothesis of serial correlation which we consider here is

$$H_3 \qquad \rho(x_i, x_{i+j}) = \rho^j, j = 1, 2, \cdots, k\,(>1)$$
$$= 0, \text{ otherwise.} \tag{15}$$

In this case, it is easily seen that

$$\epsilon(s^2) = 1 - \frac{2\rho}{n(n-1)(1-\rho)^2}[\{(n-k)(1-\rho^k)+k-1\}$$
$$\times (1-\rho) - \rho(1-\rho^{k-1})]$$
$$\cong 1, \tag{16}$$

for large $n$, and for a fixed $k$. The results below are immediate.

$$\epsilon(s) \cong 1,$$

$$\epsilon(\delta^2) \doteq 2(1-\rho), \qquad \epsilon(\delta_2{}^2) = 6 - 8\rho + 2\rho^2,$$

$$\epsilon(d) = \frac{2}{\sqrt{\pi}}(1-\rho)^{\frac{1}{2}}, \qquad \epsilon(d_2) = \sqrt{\frac{2}{\pi}}(6 - 8\rho + 2\rho^2)^{\frac{1}{2}}.$$

The expectations of the six ratio criteria under $H_3$, for large $n$, are therefore given by

$$\epsilon(w^2) \cong 2(1-\rho), \qquad \epsilon(W) \cong \frac{2}{\sqrt{\pi}}(1-\rho)^{\frac{1}{2}}$$

$$\epsilon(w_2{}^2) \cong 2(3 - 4\rho + \rho^2), \qquad \epsilon(W_2) \cong \frac{2}{\sqrt{\pi}}(3 - 4\rho + \rho^2)^{\frac{1}{2}}$$

$$\epsilon(u^2) \cong \frac{3 - 4\rho + \rho^2}{1-\rho} = 3 - \rho, \qquad \epsilon(U) \cong \sqrt{\frac{3 - 4\rho + \rho^2}{1-\rho}} = \sqrt{3-\rho} \tag{17}$$

Applying the Noéther procedure we obtain the same results as in (14). We therefore conclude that the asymptotic relative efficiencies of the six ratio criteria are the same for alternatives $H_1$ and $H_3$.

Two points may be clarified at this stage. First, although the alternatives $H_2$ and $H_3$ considered above generalise in certain directions the alternative $H_1$ considered previously by us (1962), neither $H_2$ nor $H_3$ constitutes the *complete* autoregressive scheme which is usually defined as $x_i = \rho x_{i-1} + e_i$. The reader is also referred to our previous paper (1962) in this connection. Secondly, there will be restrictions on the value of $|\rho|$ under $H_2$ or $H_3$ to ensure that the corresponding correlation matrices remain positive definite, just as we had to impose $|\rho| < \frac{1}{2}$ under $H_1$ as mentioned by us in our previous paper. It is rather difficult to determine explicitly the specific restrictions on $|\rho|$

under $H_2$ or $H_3$; but it is easy to see that the correlation matrices will remain positive definite if $|\rho|$ is small enough and this is consistent with the requirements of the Noéther procedure which assumes that the parameter $(\theta_n)$ tends to zero as $n \to \infty$ (*see* Section 2 above).

Before concluding this section we discuss the asymptotic relative efficiency under $H_2$ and $H_3$ when $k$ is less than $n$ but is itself large so that $k/n \to \lambda$. Under alternative $H_2$ we have now $\epsilon(s^2) \cong 1 - \rho(2\lambda - \lambda^2)$ and there will be corresponding modifications in the asymptotic means of $w^2$, $w_2^2$, $W$ and $W_2$ given in (11) above; but the expressions for $u^2$ and $U$ remain unaltered. Further the values of $c^{-2}$ for the first four statistics are found to be those given in (12), each multiplied by the factor $(1 - \lambda)^{-4}$. Consequently, the asymptotic relative efficiencies remain the same as before [as given in (13)]. In other words there is no change in the pattern of the asymptotic relative efficiency obtained under $H_2$, even when $k$ is large. As for $H_3$ when $k$ is large, there is no change in the expressions for the asymptotic expectations of the six criteria given in (17); and consequently there is no change in the pattern of asymptotic relative efficiency obtained above.

## 4. Alternatives of Trend

We shall now consider three alternative hypotheses of trend (i) simple linear trend, (ii) pure quadratic trend, (iii) sinusoidal trend. Here it is assumed that the $\{x_i\}$ remain independent normal but there is a trend in the mean, that is, $\epsilon(x_i) = \mu_i$ and that $\mu_i$ depends on $i$. The linear trend hypothesis will be defined by

$$H_4 \qquad \mu_i = i\alpha \tag{18}$$

where $\alpha$ is constant. It is assumed that $\alpha$ is small so that powers of $\alpha$ beyond $\alpha^2$ can be neglected. This is consistent with the Noéther procedure described in Section 2, the parameter now being $\alpha$, and the null hypothesis now being

$$H_0 \qquad \alpha = 0, \ (\mu_i = 0) \tag{19}$$

The required expectations can now be evaluated (*see e.g.*, Kamat, 1958) as follows:—

$$\begin{aligned} &\epsilon(s^2) = 1 + \frac{1}{12} n(n+1)\alpha^2, && \epsilon(s) \cong 1 + \frac{1}{24} n(n+1)\alpha^2, \\ &\epsilon(\delta^2) = 2 + \alpha^2, && \epsilon(\delta_2^2) = 6, \\ &\epsilon(d) \cong \frac{2}{\sqrt{\pi}}\{1 + \tfrac{1}{4}\alpha^2\}, && \epsilon(d_2) = \sqrt{\frac{12}{\pi}} \end{aligned} \tag{20}$$

Further let us assume $\alpha = oh(n^{-1})$. (This will be justified in the sequel.) Then, taking $\beta = \alpha n$, the hypothesis to be tested becomes $\beta = 0$ against the alternative $\beta = kn^{-\delta}$; and it will be seen that this formulation is in accordance with the Noéther procedure outlined in Section 2 above. The expectations of the first four ratio statistics under $H_4$, for large $n$, can now be written as

$$\epsilon(w^2) \cong \frac{2}{1+\frac{1}{12}\beta^2}, \qquad \epsilon(W) \cong \frac{2}{\sqrt{\pi}} \cdot \frac{1}{1+\frac{1}{24}\beta^2},$$

$$\epsilon(w_2^2) \cong \frac{6}{1+\frac{1}{12}\beta^2}, \qquad \epsilon(W_2) \cong \sqrt{\left(\frac{12}{\pi}\right)} \frac{1}{1+\frac{1}{24}\beta^2} \quad (21)$$

Applying the Noéther procedure it is seen that for these four ratios $m = 2$ and $\delta = \frac{1}{4}$. [This justifies the assumption made above, *viz.*, $\alpha = oh(n^{-1})$, since $\alpha = \beta n^{-1} = kn^{-5/4}$.] The values of $c^{-2}$ are shown to be:

$$w_2 : 36 \qquad W : 36\,(1\cdot 3058)$$

$$w_2^2 : 68 \qquad W_2 : 36\,(2\cdot 2500) \quad (22)$$

which are constant multiples of those obtained in (12) above for the alternative $H_3$. Hence the pattern of asymptotic relative efficiency is the same as in (13) for the statistics $w^2$, $w_2^2$, $W$ and $W_2$.

The remaining two criteria $u^2$ and $U$ are in a different category. From (20) we have

$$\epsilon(u^2) \cong \frac{6}{2+\alpha^2}, \qquad \epsilon(U) \cong \frac{4\sqrt{3}}{4+\alpha^2} \quad (23)$$

Using the Noéther procedure we have $m = 2$, $\delta = \frac{1}{4}$ so that the alternative tested here is $\alpha = kn^{-\frac{1}{4}}$. It follows that $u^2$ and $U$ cannot discriminate at all the hypothesis $\alpha = kn^{-5/4}$ which the other four statistics can discriminate; that is $u^2$ and $U$ have zero relative efficiency when compared to the other four. To find the asymptotic relative efficiency between the two we find $c^{-2}$ for them, which are

$$u^2 : \tfrac{2}{9} \qquad U : 0\cdot 3732 \quad (24)$$

Hence the asymptotic efficiency of $U$ (relative to $u^2$) is $0\cdot 595$ which is the same as that obtained for the alternatives $H_1$, $H_2$ or $H_3$.

We now consider the alternative of quadratic trend and for the sake of simplicity we take the pure quadratic trend denoted by

$$H_5 \qquad \mu_i = i^2 a \tag{25}$$

the null hypothesis being the same as $H_0$ given in (19). It is assumed as before that powers of $a$ beyond $a^2$ can be neglected. Proceeding in the same manner as above it is found that the six statistics form two separate groups as above, one consisting of $w^2$, $w_2^2$, $W$, $W_2$ and the other of $u^2$, $U$. (The corresponding forms of the Noéther alternatives are $a = kn^{-9/4}$ and $a = kn^{-5/4}$ respectively.) Further, it is found that in each group the pattern of the asymptotic relative efficiency is the same as obtained above for the linear trend.

The alternative of sinusoidal trend will be defined as

$$H_6 \qquad \mu_i = a \sin\left(\frac{2\pi i}{k}\right) \tag{26}$$

where $a$ is assumed to be small so that its powers after $a^2$ can be neglected, and where $k$ is assumed to be large, but it is small when compared with $n$. The hypothesis to be tested is as before $H_0$ as given in (19). Since

$$\frac{1}{n}\sum_i \mu_i^2 \backsimeq (\tfrac{1}{2})\, a^2, \quad \frac{a^{-2}}{n-1}\sum_i (\Delta\mu_i)^2 = 0\,(k^{-2}),$$

$$\frac{a^{-2}}{n-2}\sum (\Delta^2\mu_i)^2 = 0\,(k^{-2}),$$

the last two quantities can be neglected. We have therefore

$$\epsilon\,(s^2) \cong 1 + \tfrac{1}{2}a^2, \quad \epsilon\,(s) \cong 1 + \tfrac{1}{4}\, a^2 \tag{27}$$

and the expectations of $\delta^2$. $\delta_2^2$, $d$ and $d_2$ for large $n$, are approximately the same as under $H_0$. The expectations of the six ratio statistics under $H_6$, for large $n$, are therefore given by

$$\epsilon\,(w^2) \cong \frac{2}{1+\frac{1}{2}a^2}, \qquad \epsilon\,(W) \cong \frac{2}{\sqrt{\pi}}\,\frac{1}{1+\frac{1}{4}a^2}$$

$$\epsilon\,(w_2^2) \cong \frac{6}{1+\frac{1}{2}a^2}, \qquad \epsilon\,(W_2) \cong \sqrt{\frac{12}{\pi}}\,\frac{1}{1+\frac{1}{4}a^2}$$

$$\epsilon\,(u^2) \cong 3, \qquad \epsilon\,(U) \cong \sqrt{3}. \tag{28}$$

Applying the Noéther procedure we find the same pattern for the asymptotic relative efficiency under $H_6$ as that obtained under $H_2$;

that is $u^2$ and $U$ are not appropriate test procedures in this case and the asymptotic relative efficiencies of the first four are the same as that given in (13) above in Section 3.

Before concluding it may be remarked that for a trend function which is of higher order than the second it can be shown (by following the procedure of $H_4$ and $H_5$) that the pattern of the asymptotic relative efficiency is the same as the one obtained for $H_4$ or $H_5$.

## 5. Conclusions

Taking all these results together the following conclusions emerge with regard to the asymptotic relative efficiencies of the six ratio critera $w^2$, $w_2^2$, $u^2$, $W$, $W_2$ and $U$. Statistics based on squared differences, $w^2$, $w_2^2$ and $u^2$ are always more efficient than the corresponding statistics based on absolute differences, $W$, $W_2$ and $U$. The Von-Neumann statistic $w^2$ is the most efficient of the six criteria for all alternatives. For alternatives $H_1$ and $H_3$ of serial correlation the efficiency pattern is the same and the six statistics in order of their relative asymptotic efficiencies are: $w^2$, $w_2^2$, $W_2$, $W$, $u^2$, $U$. For the alternative $H_2$ of serial correlation, however, and for alternatives $H_4$, $H_5$, $H_6$ of the linear, quadratic and sinusoidal trend (and in general for a polynomial or sinusoidal trend), the six criteria are not all comparable for determining their relative asymptotic efficiency. They separate out into two groups: (i) $w^2$, $W$, $w_2^2$, $W_2$ and (ii) $u^2$, $U$. The statistics in the second group have zero relative efficiency as compared with the first. (They are either incapable of discriminating the alternative hypothesis as in $H_2$ and $H_6$ or can discriminate an alternative which is of an 'inferior' order in the Noéther sense as in $H_4$ and $H_5$.) When the criteria can discriminate the alternative hypothesis their relative asymptotic efficiencies within their particular group are the same for all these alternatives. Then their order within the group according to their relative efficiencies is that given in (i) and (ii) above.

## 6. Summary

Asymptotic relative efficiency of some test criteria for serial correlation and trend is considered under certain alternative hypotheses of serial correlation and trend.

## 7. References

Kamat, A. R. .. "On the mean successive difference and its ratio to the root mean square," *Biometrika*, 1953, **40,** 116–27.

Kamat, A. R. .. "Contributions to the theory of statistics based on the first and second successive differences," *Metron*, 1958, **19**, 97–119.

Kamat, A. R. and Sathe, Y. S. .. "Asymptotic power of certain test criteria for serial correlation between successive observations," *Ann. Math. Stat.*, 1962, **33**, 186–200.

Noéther, Gottfried .. "On a theorem of Pitman," *ibid.*, 1955, **26**, 64–69.

Tintner, G. .. *The Variate Difference Method*, Indiana, Bloomington Press, 1940.

Von Neumann, J. .. "Distribution of the ratio of the mean square successive difference to the variance," *Ann. Math. Stat.*, 1941, **12**, 367–83.