GROUP DIVISIBLE ROTATABLE DESIGNS WHICH MINIMIZE THE MEAN SQUARE BIAS

PART II : Construction

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(Received : August, 1980)

SUMMARY

Two methods of construction for GDR designs has been presented, which minimise mean square bias with respect to weight densities of a particular type.

INTRODUCTION

As stated in Part I, we are interested in those Group Divisible Rotatable (GDR) designs, which minimise the weighted mean square bias against a higher degree polynomial, with respect to a given weight density. We shall consider here, second order GDR designs, which guard against the mean square bias arising from the assumption that the actual response function is of third degree. The methods described below are for constructing designs with all odd moments of order upto fifth zeroes and with the second and fourth order moments equated to those of a given group wise spherical distribution.

We see that for m groups, the ith group consisting of n_i factors, a GDR design is specified by the following parameters, the values of which are to be given by the specific weight density assumed.

$$\lambda_2^{(i)},\ i=1, 2,\ldots, m, \quad \lambda_4^{(i)},\ i=1, 2,\ldots, m,$$

$$\theta_{ii'},\ i\neq i',\ i,\ i'=1, 2,\ldots, m.$$

Now, since these are $\binom{m}{2}$ $\theta's$, for large values of m, the task of equating $\theta's$ to a set of as many specified values may be difficult to manage. Moreover, any values of θ will not satisfy the non-singularity condition. For $m=2$, there is only one θ. In that case, the range of θ satisfying the nonsingularity conditions is as given in (4.2) in the preceeding article. The method described by Adhikari and Sinha [1] satisfies this condition, and so can be used for constructing a GDR design with $m=2$. Their method can be suitably modified in order to make the moments of the design same as those of a given group wise spherical distribution, provided those moments satisfy one more restriction, namely, $\theta=\lambda_4^{(1)}\ \lambda_4^{(2)}$. The extension of this method, to any general value of m for $\theta_{ii'}=1$, $i\neq i'$ is considered in more details in section 3.

For values of m greater than 2, the nonsingularity condition in terms of $\theta_{ii'}$'s and $\lambda_4^{(i)}$'s can be derived recursively. The parameters of the given distribution should satisfy that condition, and then it may be possible to find out method of construction for those cases, although the method may not be very easy and may include too many design points. But a simple general method not requiring a very large number of design points, *i.e.* a method applicable for any m and any form of groupwise spherical distribution is unlikely to exist, and has not been attempted in the present paper. However, when the weight density is the product of m spherical densities, we suggest two simple methods of construction, described in section 2. If furthermore, $n_1=n_2=\ldots=n_m=n$, and the weight density satisfy the condition

$$\beta_2^{(i)}=\frac{\lambda_4^{(i)}}{(\lambda_2^{(i)})^2}=\text{constant for } i=1, 2,\ldots, m.$$

we can extend the method of Adhikari and Sinha [1] with slight modification to this general case and that is discussed in section 3.

2. Method of Construction Assuming the Weight Density to be the Product of a Number of Spherical Weight Densities

Let us suppose that the given weight density is the product of m spherical densities, $m>1$.

Method 1. At first, we construct second order rotatable designs for the different groups of factors with all fifth order moments zeroes and the second and fourth order moments equated to those specified by the corresponding spherical densities, following Mukhopadhyay [7].

Let H_i be the matrix consisting of the experimental points as columns in the design constructed for the ith group $i=1, 2,\ldots, m$.

Then, $H_i=[F_i/G_i]$,

where F_i is the set of $W_i=2^{n_i-t_i}$ points of a fractional factorial $(2^{n}{}_{i})$ design of resolution VI and

$$G_i=[b_i\ I_{n_i} \mid -b_i I_{n_i} \mid c_i I_{n_i} \mid -c_i I_{n_i}]$$

Now let us partition F_i into submatrices F_{i1} and F_{i2} in such a way that each of them constitute a fraction of a 2^{n_i} experiment of resolution V. We can do this by conveniently taking F_{i1} and F_{i2} as the 1/2 replicates of F_i' other methods of doing the same thing not being ruled out.

Then we define a special product of matrices, Let $A_{m_1 \times n_1}$ and $B_{m_2 x n_2}$ be 2 matrices. Let a_i be the i-th column of A and let

$$A_i(k)=[a_i a_i \ldots a_i]_{m_1 \times k}$$

Then we define

$$A^*B=\begin{Bmatrix} A_1(n_2) & A_2(n_2) & \cdots & A_{n_1}(n_2) \\ B & B & & B \end{Bmatrix} \qquad (6.1.1)$$

Let $\quad H_{i_1}=[F_{i_1} \mid G_i]$ and $H_{i_2}=[F_{i_2} \mid G_i]$.

Then the complete set of experimental points is given by the columns of the following matrix.

$$H=[H_{1_1}{}^*\ H_{2_1} \ldots {}^*H_{m_1} \mid H_{1_2}\ {}^*\ H_{2_2} \ldots {}^*H_{m_2}]$$

For $n_i<5$, we take $H_{i_1}=H_{i_2}=H_i$

In this design, all fifth order moments are zeroes, and the second order and fourth order moments are as follows

$$\lambda_2{}^{(i)}=\frac{1}{N}\ 2 \prod_{j\neq i}(\overline{W}_j+4n_j)\ \{\overline{W}_i+2(b_i^2+c_i^2)\}$$

where $\quad \overline{W}_i=\frac{1}{2}\ .\ 2^{n_j-t_j}$

and $\quad N=2\ .\ \prod_{i=1}^{m}\ (\overline{W}_j+4n_j).$

$$\lambda_2{}^{(i)}=\frac{1}{\overline{W}_j+4n_j}\ \{\overline{W}_i+2(b_i^2+c_i^2)\} \qquad (2.1)$$

$$3\ \lambda_4{}^{(i)}=\{\overline{W}_i+2(b_i^4+c_i^4)\}/(\overline{W}_i+4ni) \qquad (2.2)$$

$$\lambda_4^{(i)}=\overline{W}_i/(\overline{W}_i+4n_i) \tag{2.3}$$

$$\theta_{ii'}=\frac{\overline{W}_i+2(b_i^2+c_i^2)}{\overline{W}_i+4n_i}\cdot\frac{W_i'+2(b_i^{2\prime}+c_i^{2\prime})}{\overline{W}_i'+4n_i'}$$

$$=\lambda_2^{(i)}\,.\,\lambda_2^{(i')}\,,\; i\neq i'$$

From (2.2) and (2.3), we get

$$b_i^4+c_i^4=2\overline{W}_i \tag{2.4}$$

Again $$\lambda_4^{(i)}/(\lambda_2^{(i)})^2=\beta_2^{(i)} \tag{2.5}$$

for the given spherical weight density.

From (2.1), we cbtain

$$2(b_i^2+c_i^2)=[\overline{W}_i(\overline{W}_i+4n_i)/\beta_2^{(i)}]^{\frac{1}{2}}-\overline{W}_i \tag{2.6}$$

solving (2.4) and (2.6), b_i^2 and c_i^2 are obtained.

Example

Let us consider an weight density which is the product of 3 circular uniform densities,

i.e. $f(x_1, x_2,\ldots, x_6)=c$

if $x_1^2+x_2^2\leqslant L^2,\ x_3^2+x_4^2\leqslant L^2,\ x_5^2+x_6^2\leqslant L^2$

$=0$, otherwise

where c is so chosen that (2.7) is a density function.

Here $$\beta_2^{(i)}=\frac{L^4/(n_i+2)\,(n_i+4)}{L^2/(n_i+2)^2}=2/3,\ i=1,2,3.$$

The set of experimental points is given by the columns of

$H=H_1{}^*H_2{}^*H_3$, where

$$H_i=\left\{\begin{array}{rrrr|rrrr|} 1 & 1 & -1 & -1 & b_i & -b_i & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & b_i & -b_i \\ & & & & c_i & -c_i & 0 & 0 \\ & & & & 0 & 0 & c_i & -c_i \end{array}\right\}$$

The approximate values of the constants are given by $b_i^2=1.985^2$ and $c_i^2=0.255^2$.

Method—2

Let F_k denote a suitable fraction (say 2^t_th) of a 2^k factorial design in which no interaction of order$\leqslant 5$ is confounded, where k is an integer and let $f_k = 2^{k-t}$.

$$\text{Let} \qquad v = b \sum_{i=1}^{m} n_i.$$

Then the experimental points may be chosen in the following manner.

1. For each pair (i, i'), we take f_v points of F_v in which all the levels of factors in the i-th group are changed to$\pm a_i$ and those in the i'-th group are changed to$\pm a_{i'}$, $i \neq i'$, the levels of the factors in other groups not being changed.

2. Let $n = \max_i n_i$.

Then we consider a GD design G_1 $(\bar{v}, b, r, k, \lambda_1, \lambda_2, m, n)$ with $\lambda_2 > \lambda_1$ and $v = mn$. We delete $n - n_i$ treatments from the i-th group of treatments, $i = 1, 2, \ldots, m$ thus reducing G_1 to another design G_2 with n_i treatments in the i-th group and unequal block sizes. Let k be the maximum block size of G_2.

Then we take f_k points of F_k corresponding to each block of G_2, treating the treatments in that block as factors, and thus obtain $b.f_k$ points in the second set by multiplying all these points by a constant d.

3. $2v$ points of the form $(0\ldots0, 0\ldots0, \ldots, 0, \pm b_i, 0\ldots0. \ldots0\ldots0)$ where b_i is the value assigned to a factor in the i-th group, $i = 1, 2, \ldots, m$.

4. $2v$ points obtained by replacing b_i in the third set by c_i.

5. $n_0 = N - \binom{m}{2} f_v - b \quad f_k - 4v$ centre points may also be included, if necessary.

The constants $a_i, b_i, b, i = 1, 2, \ldots, m$ and d are to be determined so that the group divisible relatibility conditions are satisfied.

From those conditions, we have

$$\lambda_2^{(i)} = \frac{1}{N}\left[f_v \left\{ \binom{m-1}{2} + (m-1)\, a_i^2 \right\} + r f_k\, d^3 + 2(b_i^2 + c_i^2) \right] \qquad (2.8)$$

$$3\lambda_4^{(i)}=\frac{1}{N}\left[f_v\ \{(^{m-1}_{\ 2})+(m-1)\ a_i^4\}+r\,f_k\ d^4+2\ (b_i^4+c_i^4)\ \right] \quad (2.9)$$

$$\lambda_4^{(i)}=\ \frac{1}{N}\left[\ f_v\ \{(^{m-1}_{\ 2})+(m-1)\,a_i^4\}+\lambda_1\,f_k\ d^4\ \right] \quad (2.10)$$

$$\theta_{ii'}=\frac{1}{N}\left[\ f_v\left\{\binom{m-2}{2}+(m-2)\left(\ a_i^2+\ a_{i'}^2\ \right)+a_i^2a_{\ i'}^{\ 2}\right)\right\}$$

$$+\lambda_2\ f_k\ d^4\ \Big] \quad (2.11)$$

We take $d^4=\dfrac{f_v}{\lambda_2\,f_k}\dbinom{m-1}{2}$, so that

$$\theta_{ii'}=\ \frac{f_v}{N}\left(\ a_i^2+m-2\ \right)\ \ \left(\ a_{i'}^2+m-2\ \right) \quad (2.12)$$

Now, since $\theta_{ii'}=\lambda_2^{(i)}.\lambda_2^{(i')}$

$$=(\lambda_4^{(i)}/\beta_2^{(i)}).\ (\lambda_4^{(i')}/\beta_2^{(i')})^{1/2},$$

therefore, from (2.12), we have

$$\left(\ a_i^2+m-2\ \right)^2=p_i\left\{\ a_i^4+\ \frac{m-2}{2}\,(1+\lambda_1/\lambda_2)\ \right\}$$

where $p_i=(m-1)/(\beta_2^{(i)})$,

resolving which, we get

$$a_i^2=(\ p_i-1)^{-1}\left[m-2\pm(\ p_i{}^{(m-2)}\ )^{\frac{1}{2}}\Big\{m-2-(\ ^{pi-1})/2\right.$$

$$\left.(1+\lambda_1\lambda_2)\Big\}^{\frac{1}{2}}\right] \quad (2.13)$$

From (2.9) and (2.10), we have

$$2\left(\ b_i^4+\ c_i^4\ \right)=n_v\,(m-1)\left[\ (m-2)\left(1-\ \frac{r-3\lambda_1}{2\lambda_2}\right)+2\ a_i^4\right] \quad (2.14)$$

Again, (2.8) implies

$$2(b_i^2+c_i^2)=(N.f_v)^{1/2}\ (a.\ +m-2)-f_v\binom{m-1}{2}$$

$$-(m-1)\,f_va_i^2-r\left\{\ (f_v\ f_k/\lambda_2)\ \binom{m-1}{2}\right\}^{1/2} \quad (2.15)$$

by virtue of (2.5).

Substituting the value of a_1^2 from (2.13) to (2.14) and (2.15), we get two equations in b_1^2 and c_1^2 and solve them.

Example. Let us consider the weight density given by (6.1.6). Here $v=6$, $m=3$, $n_i=2$, $\beta_2^{(i)}=2/3$, $i=1,2,3$. $f_v=2^5=32$.

We consider the following GD design with $v=6$, $b=4$, $r=2$, $k=3$, $\lambda_1=0$, $\lambda_2=1$, $m=3$, $n=2$ with treatments (1,2), (3,4), (5,6)

1	3	5
1	4	6
2	4	5

Here $f_k=8$.

The experimental points are as follows

1. $(\pm a_1, \pm a_1, \pm a_2, \pm a_2, \pm 1, \pm 1)$
$(\pm a_1, \pm a_1, \pm 1, \pm 1, \pm a_3, \pm a_3)$
$(\pm 1, \pm 1, \pm a_2, \pm a_2, \pm a_3, \pm a_3)$
96 points

2. $(\pm d, 0, \pm d, 0, \pm d, 0)$
$(\pm d, 0, 0, \pm d, 0, \pm d)$
$(0, \pm d, \pm d, 0. 0, \pm d)$
$(0, \pm d, 0, \pm d, \pm d, 0)$
32 points

3. $(\pm b_1, 0, 0, 0, 0, 0)$
$(0, \pm b_1, 0, 0, 0, 0)$
$(0, 0, \pm b_2, 0, 0, 0)$
$(0, 0, 0. \pm b_2, 0, 0)$
$(0, 0, 0, 0, \pm b_3, 0)$
$(0, 0, 0, 0, 0, \pm b_3)$
12 points

4. The fourth set is obtained by replacing b_i in the third set by c_i.

5. Centre points are not needed. The approximate values for the constants are given by

$$d^2=2,\ a_i^2=0.5,\ b_i^2=4,\ c_i^2=0.2,\ i=1,2,3.$$

6. A modification of the method described by Adhikari and Sinha (1976) to any general value of m, for the case $\theta_{ii'}=1$, $i \neq i'$ and number of factors in each group equal (say n).

Let $v=mn$,

Let us suppose that the given group wise spherical weight density is the product of m identical spherical densities, so that

$$\beta_2^{(i)}=\lambda_4^{(i)}/(\lambda_2^{(i)})^2=\text{const}=c(\text{say}),$$

i.e. $\lambda_2^{(i)}=(\lambda_4^{(i)}/c)^{1/2}$ (3.1)

Then we consider a GD design $G\ (v, b, r, k, \lambda_1\ \lambda_2, m,n)$ such that

$$c\lambda_2>\lambda_1 \text{ and } 2c\lambda_2+\lambda_1>r.$$

We consider the following sets of experimental points.

1. For each i, we take f_n points in which the levels of the factors in the ith group are same as those of F_n and the levels of factors in any other group are zeroes (f_n and F_n are as defined in method 2 of section 2), $i=1, 2,...m$.

2. The same set of points, as chosen in method 2 of section 2.

3. $2v$ points of the form

$$(0,..., 0,\pm b, 0...0)$$

4. $2v$ points of the form

$$(0,..., 0,\pm c. 0...0)$$

5. $n_0= N-m.f_n-b.f_k-4v$ centre points, if necessary.

The constants b, c and d are to be determined, so as to satisfy group divisible rotatability conditions.

From those conditions, we have

$$\lambda_2^{(i)}=N^{-1}\ (f_n+r\,f_k d^2+2\ (b^2+c^2)) \tag{3.2}$$

$$3\lambda_4^{(i)}=N^{-1}\ (f_n+r\ b_k\ d^4+2(b^2+c^2)) \tag{3.3}$$

$$\lambda_4^{(i)}=N^{-1}\ (f_n+\lambda_1 f_k\ d^4) \tag{3.4}$$

$$\theta_{ii'}=N^{-1}\ .\lambda_2 f_k\ d^4 \tag{3.5}$$

From (3.4) and (3.5), using (3.1), we have

$$\lambda_2 f_k d^4 = (f_n + \lambda_1 f_k d^4)/c$$

or $$d^4 = f_n / \{(c\lambda_2 - \lambda_1) f_k\} \quad (3.6)$$

Again, (3.3) and (3.4) implies

$$2(b^4+c^4) = 2 f_n + (3\lambda_1 - r) f_k d^4$$

$$= (2c\lambda_2 + \lambda_1 - r) f_n / (c\lambda_2 - \lambda_1)$$

using (3.6) and (3.7).

From (3.2) and (3.1), we have

$$2(b^2+c^2) = (N\lambda_2 f_k d^4)^{1/2} - f_n - r f_k d^2 \quad (3.8)$$

Solving (3.7) and (3.8) we obtain b^2 and c^2.

Example.

Let us consider the weight density which is the product of 3 circular normal weight densities, *i.e.* $m=3$ and $n=2$.

We know that, for this density, $\beta_2=3$.

Here $f_n=4, f_k=8$.

We choose the same *GD* design, as was chosen in the example of method 2. The experimental points are as follows

1.	$(\pm 1,$	$\pm 1,$	$0,$	$0,$	$0,$	$0)$
	$(0,$	$0,$	$\pm 1,$	$\pm 1,$	$0,$	$0)$
	$(0,$	$0,$	$0,$	$0,$	$\pm 1,$	$\pm 1)$

2. The same points as was chosen in method 2.
3. The columns of the matrix $[bI_6 \,|\, -bI_6]$
4. The columns of the matrix $[cI_6 \,|\, -cI_6]$
5. Central points are not needed. The value of the constants are given by

$$d^4=0.17,\ b^2=1.61,\ c^2=0.3$$

ACKNOWLEDGEMENT

The author is grateful for the help and guidance of Dr A.C. Mukhopadhyay who also suggested the problem.

REFERENCES

[1] Adhikari, B. and Sinha, B.K. (1976) : Group divisible rotatable designs. *C.S.A. Bulletin*, 79.

[2] Box, G.E.P. and Hunter, J.S. (1957) : Multifactor experimental designs for response surfaces. *A.M.S.* 28, 195.

[3] Das, M.N. and Dey, A (1967) ; Group divisible rotatable designs, *Ann. Statist. Math*, 19, 331.

[4] Draper and Lawrence, W.E. (1967) : Sequential designs for spherical weight functions. *Technometrics*, 9, 517.

[5] Herzberg, A.M. (1966) : Cylindrically rotatable designs. *A.M,S.* 37, 242.

[6] Herzberg, A.M. (1967) : Cylindrically rotatable designs *A.M.S.*, 38, 177.

[7] Mukhopadhyay, A.C. (1967) : Operability region and optimum rotatable designs. *Sankhya B* 31, 75.