A COMBINATORIAL PROBLEM AND ITS APPLICATIONS TO PROBABILITY THEORY—II

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1. INTRODUCTION

THIS paper is a continuation of previous work of the senior author on coin-tossing problems in probability theory. We shall follow closely the notations and arguments used in [1], [2] and [4].

2. STATEMENT OF PROBLEM

Let us suppose that we are given two coins 1 and 2 with probabilities p_1, p_2 of obtaining heads and consequently the probabilities q_1, q_2 of obtaining tails, where $q_i = 1 - p_i$, $i = 1, 2$. We shall assume in what follows that $p_1 + p_2 > 1$. Let us consider a game A_N, played with the following rules:—

(1) We toss coins 1 and 2 alternately, making the first trial with coin 1.

(2) We stop making further trials when the total number of heads obtained with both coins exceeds the total number of tails obtained by exactly $N\,(N \geq 1)$ for the first time.

We consider first the simpler "even" games A_{2k}, since the solution of the "odd" games A_{2k+1} is not immediate.

3. THE GAME A_{2k}, $k = 1, 2, \ldots$

The game A_{2k} can only end at the $(2n + 2k)$-th trial, $n = 0, 1, \ldots,$ and we shall have n tails and $(n + 2k)$ heads in the sequence of trials. Every sequence of A_{2k} is a series of the game g_{2k} [2] and conversely except that the probabilities are more complicated in the case of A_{2k}. The number of sequences of A_{2k} ending at the $(2n + 2k)$-th trial is therefore

$$\frac{2k}{n+2k}\binom{2n+2k-1}{n}. \tag{1}$$

We consider as in [1] and [3], the "Basic Patterns" (B.Ps.) of A_{2k}, namely, particular sequences of A_{2k} from which we can derive all possible sequences of A_{2k} at the $(2n+2k)$-th trial, by suitably inserting some (or none) of the "Subsidiary Patterns" (S.Ps.).

A B.P. of A_{2k} is a scheme formed by any series of the game g_k [2], with coin 1 and repeated with coin 2. Hence a B.P. consists only of pairs of tosses which are either

$$\begin{matrix} x \\ x \end{matrix} \quad \text{or} \quad \begin{matrix} 0 \\ 0 \end{matrix},$$

which implies that in a B.P. a head with coin 1 is followed by a head with coin 2 or a tail with coin 1 is followed by a tail with coin 2. The S.Ps. consist of the other two possible results

$$\begin{matrix} 0 \\ x \end{matrix} \quad \text{and} \quad \begin{matrix} x \\ 0 \end{matrix}.$$

For instance, in the following sequence of trials of A_4,

Coin 1	..	0	x	x	x	0	x	x
Coin 2	..	0	0	x	x	x	0	x

the B. P. is

Coin 1	..	0	x	x	x
Coin 2	..	0	x	x	x

and S.Ps. are $\begin{matrix} x \\ 0 \end{matrix}$ at 3rd and 4th trials, and $\begin{matrix} 0 & x \\ x & 0 \end{matrix}$ from 9th to 12th trials.

We divide the B.Ps. of A_{2k} into mutually exclusive classes S_r $(r = 0, 1, 2, \ldots)$ where S_r consists of all B.Ps. containing $2r$ zeros (0's). The number of B.Ps. in S_r is

$$\frac{k}{r+k}\binom{2r+k-1}{r}.$$

Evidently only the B.Ps. of $S_0, S_1, \ldots, S_m$ can enter in the formation of sequences of A_{2k} of $(2n+2k)$ trials when n is either $2m$ or $2m+1$, $m = 0, 1, \ldots$. When a particular B.P. of S_r has been chosen to yield sequences of A_{2k} which end at the $(2n+2k)$-th trial, the number of S.Ps. required is $2(m-r)$ or $2(m-r)+1$ according as $n = 2m$

or $n = 2m + 1$ respectively. The positions available for the S.Ps. in the B.P. being in both cases $(2r + k)$, a B.P. of S_r yields

$$2^{2(m-r)} \binom{2m + k - 1}{2m - 2r}$$

sequences if $n = 2m$ and

$$2^{2(m-r)+1} \binom{2m + k}{2m - 2r + 1}$$

sequences if $n = 2m + 1$. Equating the number of sequences of A_{2k} obtained from this method to the same number as given by (1) we have the identities:

$$\left.\begin{aligned} &\sum_{r=0}^{m} \frac{k}{r+k} \binom{2r + k - 1}{r} \binom{2m + k - 1}{2(m-r)} 2^{2(m-r)} \\ &\quad = \frac{k}{m+k} \binom{4m + 2k - 1}{2m}, \\ &\sum_{r=0}^{m} \frac{k}{r+k} \binom{2r + k - 1}{r} \binom{2m + k}{2(m-r)+1} 2^{2(m-r)+1} \\ &\quad = \frac{2k}{2m + 2k + 1} \binom{4m + 2k + 1}{2m + 1} \end{aligned}\right\} \quad (2)$$

for $n = 2m$ and $n = 2m + 1$ respectively.

We need to know how many of these sequences contain exactly t zeros with coin 1, so that we can assign the appropriate probability to each sequence. A simple calculation shows that the number of sequences of A_{2k} at the $(2n + 2k)$-th trial, $(n = 0, 1, 2, \ldots, k = 1, 2, \ldots)$ which come from all the B.Ps. of S_r and which contain t zeros with coin 1 is

$$[n, t, r]^k = \frac{k}{n+k} \binom{n + k - t}{r + k} \binom{n + k}{t} \binom{t}{r}. \quad (3)$$

Thus the probability that the game A_{2k} stops at the $(2n + 2k)$-th trial is

$$\sum_{r=0}^{[n/2]} \sum_{t=r}^{n-r} \frac{k}{n+k} \binom{n + k - t}{r + k} \binom{n + k}{t} \binom{t}{r} \times q_1^t q_2^{n-t} p_1^{n+k-t} p_2^{t+k} \quad (4)$$

where $[n/2]$ is the largest integer contained in $n/2$.

From our assumption, we deduce the identity

$$(p_1p_2)^k \sum_{n=0}^{\infty}\sum_{r=0}^{[n/2]}\sum_{t=r}^{n-2} \frac{k}{n+k}\binom{n+k-t}{r+k}\binom{n+k}{t}\binom{t}{r} \times (q_1p_2)^t\,(p_1q_2)^{n-t} = 1. \tag{5}$$

4. The Game A_{2k+1}, $k = 0, 1, \ldots$

A B.P. of A_{2k+1} is a scheme formed by any series of the game $g_{k+1}[2]$ with coin 1 and repeated with coin 2, omitting however the last x (head) with coin 2. We immediately see that the problem of solving A_{2k+1} is more complex by considering the fact that the S.P. $\begin{smallmatrix}x\\0\end{smallmatrix}$ cannot be placed in any position in a B.P. In trying to generate a sequence of A_1 and A_5 respectively with the given B.Ps.

$$\left|\begin{matrix}0&0&x&x\\0&0&x&x\end{matrix}\right|\begin{matrix}x\\ \end{matrix} \qquad\qquad \begin{matrix}x&x\\x&x\end{matrix}\left|\begin{matrix}0&0&x&x\\0&0&x&x\end{matrix}\right|\begin{matrix}x\\ \end{matrix}$$

it is impossible to place the S.P. $\begin{smallmatrix}x\\0\end{smallmatrix}$ in the positions indicated by the bars. (Note that the S.P. $\begin{smallmatrix}0\\x\end{smallmatrix}$ can be put in these places.) Such a position is called an "Unfavourable position" (U.P.).

We shall find, given any k ($k = 0, 1, \ldots$) and r ($r = 0, 1, \ldots$), the number of B.Ps. of S_r containing exactly b U.Ps. Let this number be denoted by $< r, b >k$ which will be obtained by using a method similar to [4] where a difference equation is constructed and solved.

Let $(r, E, b)^k$ represent for the game A_{2k+1}, the number of B.Ps. of S_r containing exactly b U.Ps. where E represents the number of heads which terminates the corresponding sequence with coin 1. We have evidently

$$(0,\ k+1,\ 1)^k = 1$$

and

$$(0,\ E,\ b)^k = 0 \quad \text{for} \quad b \neq 1,\ E \neq k+1.$$

We also obtain by recursion [4], the following difference equation:

$$\left.\begin{aligned} (r, 2, b)^k &= \sum_{u=2}^{r+k} (r-1, u, b-1)^k; \\ (r, E, b)^k &= \sum_{u=E-1}^{r+k} (r-1, u, b)^k, \\ \text{for} \quad & \\ E &= 3, 4, \ldots, r+k+1) \end{aligned}\right\} \begin{array}{l} \text{For all} \\ r = 1, 2, \ldots \\ b = 1, 2, \ldots, \\ \quad r+1. \end{array}$$

$$\begin{aligned} (1, E, b)^o &= 1 \quad \text{if} \quad E = 2 \quad \text{and} \quad b = 2 \\ &= o \quad \text{otherwise.} \\ (r, E, 1)^o &= o \quad \text{if} \quad r > o, \\ (r, E, b)^k &= o \quad \text{if} \quad r < E + b - 3 \quad \text{and} \quad E \leq r + 2, \\ (r, E, o)^k &= o, \\ (r, 2, 1)^k &= o \quad \text{if} \quad r > o. \end{aligned}$$

The solution of the difference equation is

$$(r, E, b)^k = \frac{b+E+k-4}{r+k-1} \binom{2r+k-b-E+1}{r+k-2},$$

for

$$\left.\begin{aligned} k &= 0, 1, 2, \ldots \\ r &= 0, 1, 2, \ldots \end{aligned}\right\} \text{except when } r + k = 1,$$

$$\begin{aligned} E &= 2, 3, \ldots, (r+k+1) \\ b &= 2, 3, \ldots, (r+1) \end{aligned}$$

and also for $b = 1$ when $k = 1$.

$$(r, E, 1)^k = \binom{2r+k-E}{r-1} - \binom{2r+k-E}{r+k-1}$$

for all k, r and E, except for boundary conditions mentioned above.

From the above relations we have for all $b = 1, 2, \ldots, (r+1)$; $E = 2, 3, \ldots, (r+k+1)$; $k, r = 0, 1, \ldots$ except for r, k not zero together,

$$<r, b>^k = \sum_{E=2}^{r+k+1} (r, E, b)^k = \frac{k+b-1}{r+k} \binom{2r+k-b}{r+k-1}, \tag{6}$$

$$<r, E>^k = \sum_{b=1}^{r+1} (r, E, b)^k = \binom{2r+k-E}{r-1} - \binom{2r+k-E}{r+k}, \tag{7}$$

$$<r>^k = \sum_{E=2}^{r+k+1} \sum_{b=1}^{r+1} (r, E, b)^k = \frac{k+1}{r+k+1} \binom{2r+k}{r}. \tag{8}$$

The last expression being the number of B.Ps. of S_r in A_{2k+1}.

The same technique and argument as for A_{2k} games are used to deduce the identities for A_{2k+1} games with the help of equations (6) and (8). The game will end at the $(2n + 2k + 1)$ st $(n = 0, 1, \ldots)$ trial where n can be either $2m$ or $2n + 1$. If $n = 2m$, we choose first $2(m - r) - l$ S.Ps. in the form of $\frac{x}{0}$ and place them in the favourable positions. The remaining S.Ps. in the form of $\frac{0}{x}$ are then interposed. Similarly for $n = 2m + 1$, we first place $2(m - r) - l + 1$ $\frac{x}{0}$s in the favourable positions and then insert the remaining $\frac{0}{x}$s. The total number of possible sequences of A_{2K+1} ending at the $(4m + 2k + 1)$ st trial and $(4m + 2k + 3)$ rd trial are respectively

$$\sum_{r=0}^{m} \sum_{b=1}^{r+1} \sum_{l=0}^{2(m-r)} \frac{k+b-1}{r+k} \binom{2r+k-b}{r+k-1} \binom{2m+k-b-l}{2r+k-b}$$

$$\times \binom{2m+k}{l} = \frac{2k+1}{2m+2k+1} \binom{4m+2k}{2m}, \tag{9}$$

$$\sum_{r=0}^{m} \sum_{b=1}^{r+1} \sum_{l=0}^{2(m-r)+1} \frac{k+b-1}{r+k} \binom{2r+k-b}{r+k-1} \binom{2m+k-b-l+1}{2r+k-b}$$

$$\times \binom{2m+k+1}{l} = \frac{2k+1}{2m+2k+2} \binom{4m+2k+2}{2m+1}.$$

We have then

$$[n, t, r]^k = \sum_{b=1}^{r+1} \frac{k+b-1}{r+k} \binom{2r+k-b}{r+k-1} \binom{n+k+r-t-b}{2r+k-b}$$

$$\times \binom{n+k}{t-r} \tag{10}$$

and the identity

$$p_1 (p_1 p_2)^k \sum_{n=0}^{\infty} \sum_{r=0}^{[n/2]} \sum_{t=r}^{n-r} \sum_{b=1}^{r+1} \frac{k+b-1}{r+k} \binom{2r+k-b}{r+k-1}$$

$$\times \binom{n+k+r-t-b}{2r+k-b} \binom{n+k}{t-r} (q_1 p_2)^t (p_1 q_2)^{n-t} = 1. \tag{11}$$

The relations (9), (10) and (11) are true for all k such that r and k together are not equal to zero. When $k = 0$, we have the identity

$$p_1 \left\{ \sum_{n=0}^{\infty} (q_1 p_2)^n + \sum_{n=2}^{\infty} \sum_{r=1}^{[n/2]} \sum_{t=r}^{n-r} \sum_{b=2}^{r+1} \frac{b-1}{r} \binom{2r-b}{r-1} \right.$$

$$\left. \times \binom{n+r-b-t}{2r-b} \binom{n}{t-r} (q_1 p_2)^t (p_1 q_2)^{n-t} \right\} = 1. \tag{11'}$$

5. Generating Function of A_{2k}

Let us consider a random-walk in one dimension where the particle can move two units to the right with probability p or two units to the left with probability q or stay in its position with probability r, where $p + q + r = 1$. This random-walk corresponds to the game g_k^* say, [2]· [5] which is A_{2k} if we let $p = p_1 p_2$, $q = q_1 q_2$ and $r = p_1 q_2 + p_2 q_1$.

We are interested in the probability that for the first time the particle has reached the position $+ 2k$ at the $(2n + 2k)$ th trial exactly.

From [2], the generating function of the probability distribution corresponding to the random-walk g_k^* is easily found to be

$$g_k^* (s) = \sum_{n=0}^{\infty} \frac{\frac{k}{n+k} \binom{2n+k-1}{n}}{(1 - rs)^{2n+k}} p^{n+k} q^n s^{2n+k},$$

$$= A_{2k}(s) \quad \text{where} \quad \begin{cases} p = p_1 p_2, \\ q = q_1 q_2, \\ r = p_1 q_2 + p_2 q_1. \end{cases}$$

The duration of the game, which is the expected values of the number of trials to end the game, is found to be

$$\frac{k}{p-q},$$

and the variance

$$\frac{k\,[p+q-(p-q)^2]}{(p-q)^3}.$$

SUMMARY

Games A_{2k}, A_{2k+1} arising out of tossing two coins with probabilities p_1, p_2 of obtaining heads where $p_1 + p_2 > 1$ are defined in the paper. Some of the properties of these games are derived and utilised to obtain and solve a few difference equations. The solution of the difference equations, on the other hand, leads to some interesting identities in probability theory. Besides these results, the generating function of A_{2k} is obtained from which the duration of the game and its variance are also derived.

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