ON CERTAIN METHODS OF CONSTRUCTION OF CONFOUNDED ASYMMETRICAL FACTORIAL DESIGNS WITH SMALLER NUMBER OF REPLICATIONS[1]

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I. Introduction

The factorial designs were distinguished as (i) symmetrical factorial designs and (ii) asymmetrical factorial designs according as the 'm' factors $F_1, F_2, \ldots, F_m$ in a design have (i) each the same number of levels s and (ii) levels $s_1, s_2, \ldots, s_m$ respectively where all s_i's are not equal. The problem of construction and analysis of the symmetrical factorial designs has been solved to a great extent. But not much work was done on this problem in the asymmetrical factorial designs.

Yates (1937) first introduced the asymmetrical factorial designs and obtained a number of them. This was followed by others. Nair and Rao (1941, 1942) defined the asymmetrical factorial designs and the same authors (1948) treated the problem in detail.

Recently several methods of construction of confounded asymmetrical factorial designs have been evolved through the works of Kishen and Srivastava (1959), Das (1960) and others. Kishen and Srivastava obtained the asymmetrical factorial designs of the type $q \times S^n$ in blocks of size $q \times S^k$, where $q < S$ and S is a prime power, through surfaces defined in asymmetrical factorial designs. Das obtained designs of the type $q_1 \times q_2 \times \ldots \times q_r \times S^n$ in blocks of size $q_1 \times q_2 \times \ldots \times q_r \times S^k$, where q_i $(i = 1, 2, \ldots, r)$ can be any number, as fractional replications of some corresponding symmetrical factorical designs. Both Kishen and Das advanced further methods of construction of balanced asymmetrical factorial designs of the type $q \times 2^2$ in $2q$ plot blocks by making use of the properties of balanced incomplete block designs.

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In the present paper several methods of construction of asymmetrical factorial designs of the type $q \times 2^n$ in $q \times 2^p$ plot blocks through incomplete block designs are presented. These designs have the property that they can be obtained with smaller number of replications without any loss of precision on this account.

2. Designs of the Type $q \times 2^2$ in $2q$ Plot Blocks through P.B.I.B. Designs

We shall first construct designs of the type $q \times 2^2$ in $2q$ plot blocks for obtaining the more general class of designs.

2.1. *In* b *Replications.*

Let X, A, and B be the three factors at levels q, 2 and 2 respectively. The interaction AB can be expressed as the contrast $(\alpha - \beta)$ where α stands for the combinations (00 and 11) and β for the combinations (01 and 10) of A and B.

First we obtain a partially balanced incomplete block design, with the q levels of the factor X as the treatments, in b blocks each of size, say, k with r replications. Two blocks will be generated from each of the blocks of the P.B.I.B. design. Let us take, say, the i-th block of the P.B.I.B. design. By associating each of the two treatment combinations in α with all the treatments (levels of X) in the i-th block of P.B.I.B. design and then by associating each of the two treatment combinations in β with all the treatments (levels of X) that are not occurring in the i-th block of the P.B.I.B. design, we generate the first block (i 1) of the design. By interchanging α and β in (i 1) we generate the second block (i 2). Thus, by generating two blocks from each of the b blocks of the P.B.I.B. design we construct the asymmetrical factorial design $q \times 2^2$ in $2b$ blocks, each of size $2q$. It can be seen that the number of replications in this design is b.

Analysis.—It can be shown through least square principles applied for the estimation of constants in factorial model that the effects of the two affected interactions AB and XAB are estimable mutually independently. The estimates of the effects of the unaffected interactions and the $S.S.$ due to them can be obtained in the usual manner.

For estimating the effects of the two factor interaction AB, we define effects t_α and t_β due to it, where $t_\alpha = (ab)_{00} = (ab)_{11}$ and $t_\beta = (ab)_{01} = (ab)_{10}$, $(ab)_{jk}$ being the AB interaction effect in the factorial model assumed. The normal equations for estimating t_α and t_β after eliminating the block effects come out as:

$$P_\alpha = 2bqt_\alpha - \frac{4b\{k^2 + (q-k)^2\}}{2q}\, t_\alpha - \frac{8bk\,(q-k)}{2q}\, t_\beta$$

$$P_\beta = 2bq\, t_\beta - \frac{4b\{k^2 + (q-k)^2\}}{2q}\, t_\beta - \frac{8bk\,(q-k)}{2q}\, t_\alpha \qquad (1)$$

where

$$P_\alpha = (AB)_0 - \left(\frac{k}{q}\right)\sum_i B_{i1} - \frac{(q-k)}{q}\sum_i B_{i2}$$

$$P_\beta = (AB)_1 - \left(\frac{k}{q}\right)\sum_i B_{i2} - \frac{(q-k)}{q}\sum_i B_{i1}$$

$(AB)_0$ is the total of all observations in the design involving either 00 or 11 combinations of A and B.

$(AB)_1$ has similar meaning as $(AB)_0$

and B_{ij} stands for the (ij)-th block total for $i = 1, 2, \ldots, b$ and $j = 1, 2$.

With the restriction $t_\alpha + t_\beta = 0$ the Equations (1) become

$$P_\alpha = \left\{2bq - \frac{4b\,(q-2k)^2}{2q}\right\} t_\alpha$$

$$P_\beta = \left\{2bq - \frac{4b\,(q-2k)^2}{2q}\right\} t_\beta \qquad (2)$$

Thus the *S.S.* due to the interaction AB is

$$\frac{P_\alpha^2 + P_\beta^2}{2bq - \dfrac{4b\,(q-2k)^2}{2q}}.$$

The estimate of the contrast $(t_\alpha - t_\beta)$ of the interaction AB is

$$\frac{P_\alpha - P_\beta}{2bq\left\{1 - \dfrac{(q-2k)^2}{q^2}\right\}}$$

and its variance V is

$$\frac{2\sigma^2}{2bq\left\{1 - \dfrac{(q-2k)^2}{q^2}\right\}},$$

where σ^2 is the error variance.

If the design is such that the interaction AB is not confounded the variance V' of the estimate of the same contrast $(t_\alpha - t_\beta)$ would be $2\sigma^2/2bq$.

Hence the relative loss of information on AB in the design,

$$1 - \left(\frac{V'}{V}\right) = \left\{1 - \left(\frac{2k}{q}\right)\right\}^2.$$

It will be seen that if q is even and k is chosen to be equal to $q/2$, the loss of information on AB becomes zero and hence the interaction AB will be unconfounded in the design.

The normal equations for estimating the effects of the interaction XAB after eliminating the block effects come out as:

$$Q_i = \left(4b - \frac{8b}{2q}\right)t_i - \sum_m \frac{8\{2(b - 2r + 2\lambda_m) - b\}}{2q} T_{im}$$

for $i = 0, 1, 2, \ldots, (q-1)$, (3)

where

$$Q_i = T_i - \left(\frac{1}{q}\right) \sum_{j=1}^{b} \sum_{k=1}^{2} \delta_{jkr} B_{jk}$$

$t_i = (xab)_{i00} = (xab)_{i11} = -(xab)_{i01} = -(xab)_{i10}$

$(xab)_{ijk}$ = the XAB interaction effect in the factorial model assumed.

δ_{jkr} is $+1$ or -1 according as x, the r-th level of x, occurs in the jk-th block with α or β.

T_{im} denotes the sum of all the t_j's that involve those x_j's, which are the m-th associates of x_i in the P.B.I.B. design,

T_i = (sum of all observations involving x_i00 or x_i11) $-$ (sum of all observations involving x_i01 or x_i10)

and b, q, r, k and λ_m are the constants of the P.B.I.B. design used to construct the design $q \times 2^2$.

The method of obtaining the solutions of these equations in t_i's corresponds exactly to that of the effects x_i's in the case of the P.B.I.B. design used. The different contrasts among t_i's are estimated with different precisions and the design can be called partially balanced asymmetrical factorial design.

The *S.S.* due to *XAB* can be obtained as $\Sigma\, t_i Q_i - (\Sigma\, t_i)\,.\,(\Sigma Q_i)/q$.

(*Note:* ΣQ_i, in this case, is not equal to zero.)

2.2. *In* b-*half Replications.*

Instead of taking $2b$ blocks as stated earlier, if we take only one set of b blocks, *i.e.*, the blocks numbered (11, 21, 31, ..., b1) or (12, 22, 32, ..., b2), we can get the design $q \times 2^2$ in b blocks of $2q$ plots, *i.e.*, with only b-half replications. (Each of the blocks is a half replication.) The normal equations in these designs, when blocks (11, 21, 31, ..., b1) are used, for estimating the effects of the interaction AB will be as below:

$$P_\alpha = \left(2bk - \frac{4bk^2}{2q}\right) t_\alpha - \frac{4bk\,(q-k)}{2q}\, t_\beta$$

$$P_\beta = \left\{2b\,(q-k) - \frac{4b\,(q-k)^2}{2q}\right\} t_\beta - \frac{4bk\,(q-k)}{2q}\, t_\alpha \qquad (4)$$

where

$$P_\alpha = (AB)_0 - \left(\frac{k}{q}\right) \sum_i B_{i1}$$

$$P_\beta = (AB)_1 - \frac{(q-k)}{q} \sum_i B_{i1}$$

$(AB)_0$ and $(AB)_1$ have the same meaning as in Section 2.1.

Taking the restriction $kt_\alpha + (q-k)\, t_\beta = 0$, we have

$$P_\alpha = 2bkt_\alpha; \text{ and } P_\beta = 2b\,(q-k)\, t_\beta \qquad (5)$$

Thus the *S.S.* due to *AB* is

$$\frac{P_\alpha^{\,2}}{2bk} + \frac{P_\beta^{\,2}}{2b\,(q-k)}\,.$$

The estimate of the contrast $(t_\alpha - t_\beta)$, of the interaction AB, is $(\tfrac{1}{2}b)\,\{(P_\alpha/k) - (P_\beta/\overline{q-k})\}$ with variance $q\sigma^2/2bk\,(q-k)$. If the interaction AB was not confounded the variance would have been $2\sigma^2/bq$. Hence the relative loss of information on AB is $(1 - 2k/q)^2$, which is the same as in the case of designs obtained in Section 2.1 with b replications.

The normal equations for estimating the effects of the interaction XAB after eliminating block effects come out as:

$$Q_i = \left(2b - \frac{4b}{2q}\right) t_i - \sum_m \frac{4\{2(b - 2r + 2\lambda_m) - b\}}{2q} T_{im}$$

for $i = 0, 1, 2, \ldots, (q - 1)$. (6)

The estimates of the effects t_i's and the $S.S.$ due to them can be obtained as in Section 2.1.

It is to be added here that these designs in b-half replications, in general, need neither be equi-replicated nor resolvable. These have a special significance because of the fact that the estimates of the effects of all the interactions can be obtained mutually independently and also the relative loss of information on the contrasts of the affected interactions remains the same, as in the case of designs in b replications. However, when q is even and k is chosen to be $q/2$, we have $b = 2r$ and the design will then be an equi-replicated one. In this case interaction AB will be unconfounded and information is lost only on the interaction XAB.

3. Some Special Cases

We shall here consider the two cases of designs obtained through (i) a B.I.B. design and (ii) a group-divisible design.

In either of these cases the normal equations for the estimation of the effects of the interaction AB are unchanged and the relative loss of information on AB is the same as $\{1 - (2k/q)\}^2$. The normal equations for the estimation of the effects of the interaction XAB in these cases will be particular cases of Equations (3) above. We shall hence consider the normal equations for the estimation of these effects of XAB, in these two cases (for designs in b replications), and will solve them to obtain the $S.S.$ and the losses of information on different contrasts of the interaction.

3.1. *Using a B.I.B. Design.*

The normal equations for the estimation of the effects of the interaction XAB, under the restriction

$\sum_j t_j =$ a constant, come out as

$$Q_i = \left\{4b - \frac{32(r - \lambda)}{2q}\right\} t_i + \text{a constant}$$

for $i = 0, 1, 2, \ldots, (q - 1)$ (7)

The relative loss of information on any contrast $\Sigma l_i t_i$ of the interaction XAB is $4(r-\lambda)/bq = 4k(q-k)/[q^2(q-1)]$. Hence the design would be a balanced one. The total loss of information in the design $= (1-2k/q)^2 x_1 + 4k(q-k)/q^2(q-1) \times (q-1) = 1$ = Number of blocks per replication less by one.

3.2. *Using a Group-divisible Design.*

Let the group-divisible design be with m groups, each of n elements, *i.e.*, $q = mn$. Then the normal equations for the estimation of the effects of XAB under the restriction $\Sigma t_i =$ a constant, are

$$Q_i = \left\{4b - 32\frac{(r-\lambda_1)}{2q}\right\} t_i - \left(\frac{32}{2q}\right)(\lambda_1-\lambda_2)\, T_{i1}' + \text{a constant}$$

$$\text{for } i = 0, 1, 2, \ldots, (q-1). \qquad (8)$$

where

$$T_{i1}' = t_i + T_{i1}$$

Solving Equations (8) for t_i's we get

$$t_i = \frac{Q_i}{4b - \dfrac{32(r-\lambda_1)}{2q}} + \frac{32(\lambda_1-\lambda_2)\,Q_{i1}'}{2q\left\{4b - \dfrac{32(r-\lambda_1)}{2q} - \dfrac{32n(\lambda_1-\lambda_2)}{2q}\right\}\left\{4b - \dfrac{32(r-\lambda_1)}{2q}\right\}} + \text{a constant}$$

$$\text{for } i = 0, 1, 2, \ldots, (q-1). \qquad (9)$$

where Q_{i1}' is the sum of Q_i's similar to T_{i1}' of t_i's.

It can be shown easily that the $(q-1)$ orthogonal, normalised, linear, independent contrasts of XAB are:

I. the $m(n-1)$ contrasts $\sum_{r=1}^{n} L_{jr} t_{i_r}$

$$\text{for } j = 1, 2, \ldots, (n-1) \text{ and } i = 1, 2, \ldots, m.$$

each with relative loss of information $4(r-\lambda_1)/bq$, where $i1, i2, \ldots in$ are the n treatments forming the i-th group of the Group-divisible design.

II. the $(m-1)$ contrasts $\sum_{v=1}^{m} L_{uv} T_v$ for $u = 1, 2, \ldots, (m-1)$, each with relative loss of information

$$\frac{4(r-\lambda_1)+4n(\lambda_1-\lambda_2)}{bq}$$

where $T_v = \sum_{r=1}^{n} t_{vr}$.

Thus the total loss of information in the design = 1 = Number of blocks per replication less by one. These results also hold good in the case of designs in b-half replications.

4. Designs of the Type $q \times 2^n$ in $q \times 2^p$ Plot Blocks

Let $X, A_1, A_2, \ldots, A_n$ be the factors at levels $q, 2, 2, \ldots, 2$ respectively. We shall first obtain D_1, a design 2^n in 2^{p+1} plot blocks with $A_1, A_2, \ldots, A_n$ as the factors. Let the interactions confounded between blocks in D_1 be called as "the between block set of interactions". We then divide each of the blocks of D_1 into two sub-blocks by confounding one more independent interaction, say I, and hence 2^{n-p-1} more interactions between sub-blocks. Let these 2^{n-p-1} interactions be called as "the between sub-blocks within block set of interactions". We shall denote by α_j all the 2^p treatment combinations in the j-th block of D_1 which have an even number of letters common with the interaction I and form one of the two sub-blocks of the j-th block of D_1. The remaining 2^p treatment combinations, forming the other sub-block of the j-th block of D_1, will be denoted by β_j.

Now, as in the case of design $q \times 2^2$ in $2q$ plot blocks, we obtain a P.B.I.B. design D_2 with the treatments in b blocks each of size k. By combining this P.B.I.B. design D_2 with D_1 we construct the design $q \times 2^n$ in $q \times 2^p$ plot blocks.

4.1. *In* b *Replications.*

We combine one block, say i-th, of the design D_2 with one block, say j-th, of the design D_1 to generate two blocks of the asymmetrical factorial design by a method similar to the one used in Section 2.1 above. By associating each of the treatment combinations in α_j with all the treatments (levels of X) occurring in the i-th block of D_2 and by associating each of the treatment combinations in β_j with all those treatments that are not occurring in the i-th block of D_2 the ($ij1$)-th block of the design is generated. By interchanging α_j and β_j in ($ij1$) the block ($ij2$) will be generated.

Thus by combining each of the b blocks of D_2 with each of the 2^{n-p-1} blocks of D_1 we generate $b.2^{n-p}$ blocks of the asymmetrical factorial design $q \times 2^n$ in $q \times 2^p$ plot blocks. The number of replications in this design will be b.

4.2. *In* b-*half Replications.*

As in the case of the designs $q \times 2^2$ in $2q$ plot blocks we can obtain the design $q \times 2^n$ in $q \times 2^p$ plot blocks with only b-half replications by generating only one block ($ij1$) of it from every combination of the i-th block of D_2 and j-th block of D_1. The remaining set of blocks ($ij2$) also forms a design $q \times 2^n$ in $q \times 2^p$ plot blocks with b-half replications.

In these designs, with b or b-half replications, the different interactions affected are (i) all the interactions belonging to the between block set, (ii) all the interactions of the type U, belonging to the between sub-blocks within block set and (iii) all the interactions of the type XU.

All the interactions in (i) above are confounded between blocks of the design and no information is available on them.

By writing down the normal equations, it can be shown that each of the interactions in (ii) above loses $(1 - 2k/q)^2$ relative information, and that the different contrasts of interactions XU can be estimated, but with different precisions. If, however, we use a B.I.B. design for the construction, the relative loss of information on the interaction XU will be $4k(q-k)/q^2(q-1)$. The total loss of information is $2^{n-p} - 1$ = Number of blocks per replication less by one.

The estimates of the effects of the affected interactions and their $S.S.$ can be obtained as in the case of the design $q \times 2^2$ in $2q$ plot blocks. (The unaffected interactions do not create any complications.) It can be shown that the effects of the different affected interactions can be obtained mutually independently.

5. Designs of the Type $q \times 2^2$ in $2q$ Plot Blocks in Two Replications, when q is Even

As before, we shall first consider the case of construction of the designs of the type $q \times 2^2$ in $2q$ plot blocks. The generalisation has subsequently been indicated in Section 6. These designs in two replications are constructed only when q is even, *i.e.*, $q = 2m$, say.

Let X, A and B be the three factors at levels q, 2 and 2 respectively. The $2m$ levels of the factor X are formed into four groups G_1, G_2, G_3 and G_4 of sizes l, $(m - l)$, l and $(m - l)$ respectively. Let I_j denote level of the factor X for $j = 1, 2, \ldots, 2m$ such that $I_j \neq I_j'$, for $j \neq j'$. Let $I_1, I_2, \ldots, I_l$ be the l levels of X in G_1; $I_{l+1}, I_{l+2}, \ldots, I_m$ be the $(m - l)$ levels in G_2; $I_{m+1}, I_{m+2}, \ldots, I_{m+l}$ the l levels in G_3; and $I_{m+l+1}, I_{m+l+2}, \ldots, I_{2m}$ the $(m - l)$ levels in G_4.

The following incomplete block design with $q = 2m$ levels of X as treatments in four blocks, each of size m, with 2 replications will be used to construct the design $2m \times 2^2$ in $4m$ plot blocks with 2 replications. The incomplete block design is:

Block No.	Treatments			
1	I_1	$I_2 \ldots I_l$	I_{l+1}	$I_{l+2} \ldots I_m$
2	I_{m+1}	$I_{m+2} \ldots I_{m+l}$	I_{m+l+1}	$I_{m+l+2} \ldots I_{2m}$
3	I_1	$I_2 \ldots I_l$	I_{m+1+1}	$I_{m+l+2} \ldots I_{2m}$
4	I_{l+1}	$I_{l+2} \ldots I_m$	I_{m+1}	$I_{m+2} \ldots I_{m+l}$

Note.—The contents of 1st block are taken to be all the $l + (m - l) = m$ treatments in G_1 and G_2. The contents of the 2nd block are all the m treatments in G_3 and G_4. By replacing the treatments of G_2 in the 1st block by those of G_4, the 3rd block is obtained. Similarly by replacing the treatments of G_4 in the 2nd block by those of G_2, the 4th block is obtained.

From this incomplete block design, we obtain the design $q \times 2^2$ in $2q$ plot blocks, with 2 replications by generating only one block of the asymmetrical factorial design from each block of the incomplete block design, by associating the treatment combinations (00, 11) of A and B in α and (01, 10) in β with the treatments of the incomplete block design as in Section 2.2. The asymmetrical factorial design obtained will be as under:

Block No.	Treatment combinations					
1	$I_1\alpha$,	$I_2\alpha$,	$\ldots, I_l\alpha$,	$I_{l+1}\alpha$,	$I_{l+2}\alpha$,	$\ldots, I_m\alpha$,
	$I_{m+1}\beta$,	$I_{m+2}\beta$,	$\ldots, I_{m+l}\beta$,	$I_{m+l+1}\beta$,	$I_{m+l+2}\beta$,	$\ldots, I_{2m}\beta$.
2	$I_1\beta$,	$I_2\beta$,	$\ldots, I_l\beta$,	$I_{l+1}\beta$,	$I_{l+2}\beta$,	$\ldots, I_m\beta$,
	$I_{m+1}\alpha$,	$I_{m+2}\alpha$,	$\ldots, I_{m+l}\alpha$,	$I_{m+l+1}\alpha$,	$I_{m+l+2}\alpha$,	$\ldots, I_{2m}\alpha$.
3	$I_1\alpha$,	$I_2\alpha$,	$\ldots, I_l\alpha$,	$I_{l+1}\beta$,	$I_{l+2}\beta$,	$\ldots, I_m\beta$,
	$I_{m+1}\beta$,	$I_{m+2}\beta$,	$\ldots, I_{m+l}\beta$,	$I_{m+l+1}\alpha$,	$I_{m+l+2}\alpha$,	$\ldots, I_{2m}\alpha$.
4	$I_1\beta$,	$I_2\beta$,	$\ldots, I_l\beta$,	$I_{l+1}\alpha$,	$I_{l+2}\alpha$,	$\ldots, I\alpha_m$,
	$I_{m+1}\alpha$,	$I_{m+2}\alpha$,	$\ldots, I_{m+l}\alpha$,	$I_{m+l+1}\beta$,	$I_{m+l+2}\beta$,	$\ldots, I_{2m}\beta$.

where $I_j\alpha$ stands for the two combinations I_j00 and I_j11 and $I_j\beta$ for the combinations I_j01 and I_j10 of the factors X, A and B.

In this design the interaction XAB alone is affected. The estimates of the unaffected interaction effects and their $S.S.$ can be obtained in the usual manner.

We shall obtain the normal equations for the estimation of the effects of the interaction XAB after eliminating blocks effects. For this purpose we shall define the XAB interaction effects, t_j's for $j = 1, 2, \ldots, 2m$ where $t_j = (xab)_{ij00} = (xab)_{ij11} = -(xab)_{ij01} = -(xab)_{ij10}$. The normal equations are

$$Q_{1i} = 8\,t_{1i} - \left(\frac{16}{4m}\right)\sum_j t_{1j} + \left(\frac{16}{4m}\right)\sum_j t_{3j} \quad \text{for} \quad i = 1, 2, \ldots, l.$$

$$Q_{2i} = 8\,t_{2i} - \left(\frac{16}{4m}\right)\sum_j t_{2j} + \left(\frac{16}{4m}\right)\sum_j t_{4i} \quad \text{for} \quad i = 1, 2, \ldots, (m - l).$$

$$Q_{3i} = 8\,t_{3i} - \left(\frac{16}{4m}\right)\sum t_{3j} + \left(\frac{16}{4m}\right)\sum_j t_{1j} \quad \text{for} \quad i = 1, 2, \ldots, l.$$

$$Q_{4i} = 8\, t_{4i} - \left(\frac{16}{4m}\right) \sum_j t_{4j} + \left(\frac{16}{4m}\right) \sum_j t_{2j}$$

$$\text{for} \quad i = 1,\ 2, \ldots, (m - l) \qquad (10)$$

where

$t_{1i} = t_i$ in the original nomenclature;

$t_{2i} = t_{l+i}$; $t_{3i} = t_{m+i}$; $t_{4i} = t_{m+l+i}$;

$\sum_j t_{1j} = t_1 + t_2 + \ldots + t_l$;

$\sum_j t_{2j} = t_{l+1} + t_{l+2} + \ldots + t_m$;

$\sum_j t_{3j} = t_{m+1} + t_{m+2} + \ldots + t_{m+l}$;

$\sum_j t_{4j} = t_{m+l+1} + t_{m+l+2} + \ldots + t_{2m}$;

Q_{1i}, Q_{2i}, Q_{3i} and Q_{4i} are defined similar to $t_{1i}, t_{2i}, t_{3i}, t_{4i}$ respectively.

$Q_j = T_j - (\frac{1}{2}m)\, \delta_j$.

T_j = (Sum of all observations due to the two combinations I_j00, I_j11 of X, A and B) – (Sum of all observations due to the two combinations I_j01 and I_j10).

δ_j, the adjustment for block effects

$= B_1 - B_2 + B_3 - B_4$ for $j = 1,\ 2, \ldots, l$

$= B_1 - B_2 - B_3 + B_4$ for $j = l + 1,\ l + 2, \ldots, m$

$= - B_1 + B_2 - B_3 + B_4$ for $j = m + 1,\ m + 2, \ldots, m+l$

$= - B_1 + B_2 + B_3 - B_4$ for $j = m + l + 1, m + l + 2, \ldots, 2m$

and B_i is the total of observations in the i-th block.

Solving the normal Equations (10) for t_i's we get

$$t_{1i} = \frac{1}{8}\, Q_{1i} + \frac{1}{16\,(m - l)} \left(\sum_j Q_{1j} - \sum_j Q_{3j} \right)$$

$$\text{for} \quad i = 1,\ 2, \ldots, l$$

$$t_{2i} = \frac{1}{8}\, Q_{2i} + \frac{1}{16l} \left(\sum_j Q_{2j} - \sum_j Q_{4j} \right)$$

$$\text{for} \quad i = 1,\ 2, \ldots, (m - l)$$

$$t_{3i} = \frac{1}{8} Q_{3i} - \frac{1}{16(m-l)} \left(\sum_j Q_{1j} - \sum_j Q_{3j} \right)$$

for $i = 1, 2, \ldots, l$

$$t_{4i} = \frac{1}{8} Q_{4i} - \frac{1}{16l} \left(\sum_j Q_{2j} - \sum_j Q_{4j} \right)$$

for $i = 1, 2, \ldots, (m-l)$ (11)

where $\sum_j Q_{rj}$ is defined similar to $\sum_j t_{rj}$ for $r = 1, 2, 3$ and 4.

The $(2m-1)$ linear, independent, normalised, orthogonal contrasts between t_i's corresponding to the interaction XAB are:

I. the $(l-1)$ contrasts $\sum_{s=1}^{l} p_{rs} t_{1s}$ for $r = 1, 2, \ldots, (l-1)$.

II. the $(m-l-1)$ contrasts $\sum_{j=1}^{m-l} p_{ij} t_{2j}$ for $i=1, 2, \ldots, (m-l-1)$

III. the $(l-1)$ contrasts $\sum_{s=1}^{l} p_{rs} t_{3s}$ for $= 1, 2, \ldots, (l-1)$

IV. the $(m-l-1)$ contrasts $\sum_{j=1}^{m-l} p_{ij} t_{4j}$ for $i = 1, 2, \ldots, (m-l-1)$

V. the contrasts $\frac{1}{\sqrt{2l}} \left(\sum_j t_{1j} - \sum_j t_{3j} \right)$

VI. the contrast $\frac{1}{\sqrt{2(m-l)}} \left(\sum_j t_{2j} - \sum_j t_{4j} \right)$

and

VII. the contrast $\frac{m-l}{\sqrt{2ml(m-l)}} \left(\sum_j t_{1j} + \sum_j t_{3j} \right)$

$$- \frac{l}{\sqrt{2ml(m-l)}} \left(\sum_j t_{2j} + \sum_g t_{4j} \right).$$

The $S.S.$ due to XAB can be obtained in the usual manner as

$$\sum t_i Q_i - \frac{(\Sigma\, t_i)\,.\,(\Sigma\, Q_i)}{q}.$$

For the practical convenience the $S.S.$ with $(2m-3)$ d.f. of XAB corresponding to the contrasts in I, II, III, IV and VII above can be obtained as

$$\left(\frac{1}{8}\right)\sum T_i^{\,2} - \frac{(\Sigma\, T_i)^2}{16m} - \frac{\left(\sum_j T_{1j} - \sum_j T_{3j}\right)^2}{16l}$$
$$- \frac{\left(\sum_j T_{2j} - \sum_j T_{4j}\right)^2}{16\,(m-l)}$$

where T_{rj}'s are defined similar to Q_{rj}'s for $r = 1, 2, 3$ and 4. The $S.S.$ due to 1 d.f. of XAB corresponding to the contrast in V can be obtained from the estimate of the contrast as

$$\frac{m\left(\sum_j Q_{1j} - \sum_j Q_{3j}\right)^2}{16l\,(m-l)}$$

and the $S.S.$ due to remaining 1 d.f. corresponding to the contrast in VI can be obtained as

$$\frac{m\left(\sum_j Q_{2j} - \sum_j Q_{4j}\right)^2}{16l\,(m-l)}.$$

The contrasts in I, II, III, IV and VII above, each with a variance $\sigma^2/8$ are unaffected in the design and no information is lost on them. The variance of the estimate of the contrast in V is equal to $m\sigma^2/8\,(m-l)$. So the relative loss of information on this contrast is l/m. Similarly, the relative loss of information on the contrast in VI is $(m-l)/m$.

Thus these designs have only 2 d.f. of the interaction XAB affected with the relative losses of information on them as l/m and $(m-l)/m$. The total loss of information in the design $l/m + (m-l)/m = 1 =$ Number of blocks per replication less by one.

6. Designs of the Type $q \times 2^n$ in $q \times 2^p$ Plot Blocks in Two Replications, when q is even

Let $X, A_1, A_2, \ldots, A_n$ be the $(n+1)$ factors at levels $q, 2, 2, \ldots, 2$ respectively. Firstly we obtain the design D_1, *i.e.*, $(2^n, 2^{n-p-1})$ with

$A_1, A_2, \ldots, A_n$ as the n factors having each of its blocks divided into two sub-blocks, as in Section 4. Using the same definitions for α_j and β_j we combine D_1 with the incomplete block design given in Section 5, and generate 2^{n-p+1} blocks of the asymmetrical factorial design, each of which is of size $q \times 2^p$, by the method given in Section 4.2. The number of replications in this design is two.

Using the definitions for the "between block set of interactions" and "the between sub-block within block set of interactions", given in Section 4, it can be seen that in this design (i) all the interactions belonging to the between block set are completely confounded between blocks and hence no information is available on them and (ii) all the interactions of the type XU, where U is any interaction belonging to the between sub-blocks within block set are also affected.

Defining $t_j(U)$ as the j-th effect of the interaction XU, it can be shown that for any given U only two d.f. of the interaction XU are affected. These two d.f. correspond to the two contrasts of XU, *viz.*,

$$\text{(i)} \quad \frac{1}{\sqrt{2l}}\left\{\sum_j t_{1j}(U) - \sum_j t_{3j}(U)\right\}$$

$$\text{(ii)} \quad \frac{1}{\sqrt{2(m-l)}}\left\{\sum_j t_{2j}(U) - \sum_j t_{4j}(U)\right\}$$

with the relative losses of informations l/m and $(m-l)/m$ respectively. The total loss of information in this design is $2^{n-p} - 1$ = Number of blocks per replication less by one.

7. Summary

In this paper methods of constructing designs of the type $q \times 2^n$ in $q \times 2^p$ plot blocks through P.B.I.B. designs in (i) b replications, and (ii) b-half replications, where b is the number of blocks of the P.B.I.B. design, have been described. Another method through which such designs can be obtained in two replications, when q is even, has also been presented. The different interactions affected together with their losses of information, in all these designs have been indicated along with the method of analysis.

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