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A Class of Estimators for Mean of Symmetrical Population when the Variance is not known

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SUMMARY

A class of estimators of population mean (μ) when the variance (σ^2) is unknown, is proposed in case of symmetrical populations. Bias and mean squared error are found for the class. Various estimators are shown to belong to the class and sub-class of optimum estimators in the sense of having minimum mean squared error is found.

Key words : Class of estimators, Coefficient of variation, Mean square error, Unknown variance.

Introduction

Utilising known square of coefficient of variation $C^2\left(=\frac{\sigma^2}{\mu^2}\right)$, Searles [2] proposed an improved estimator of population mean μ; but when C^2 is unknown, the problem of estimation consists of estimators using the estimates of C^2 given by

$$\hat{C}^2=\frac{s^2}{\bar{y}^2} \text{ or } \hat{C}^2=\frac{s^2}{\bar{y}^2}\left(1-\frac{s^2}{n\,\bar{y}^2}\right)^{-1}$$

where $\bar{y}=\frac{1}{n}\sum_{i=1}^{n} y_i$ and $s^2=\frac{1}{(n-1)}\sum_{i=1}^{n}(y_i-\bar{y})^2$ for the values $y_1, y_2, ..., y_n$ of a random sample of size n.

In this paper, with $u=\frac{s^2}{n\,\bar{y}^2}$, the following class of estimators are proposed for population mean μ

$$t=f\left(\bar{y}, \frac{s^2}{n\,\bar{y}^2}\right)=f(\bar{y}, u)$$

where $f(\bar{y}, u)$ satisfying the validity conditions of Taylor's series expansion, is the function of $(\bar{y}, u)$ such that $f(\mu, 0)=\mu$, first order partial derivative

REFERENCES

[1] Greenberg, B.G., Kuebler, Roy R. Jr., Abernathy, James R. and Hortvitz Daniel G., 1971. Application of the randomized response technique in obtaining quantitative data. *J. Amer. Statist. Assoc.* **66.** 243-250.

[2] Horvitz, Daniel G., Greenberg, B.G. and Abernathy, J.R., 1975. Recent development in randomized response designs. A survey of statistical Design and linear model. Edited by J.N. Srivastava, North Holland Publishing Company, Amsterdam, 271-285.

[3] Moors, J.J.A., 1971. Optimization of the unrelated question randomized response model. *J. Amer. Statist. Assoc.* **66,** 627-629.

[4] Warner, Stanley L., 1965. Randomized response : A survey technique for eliminating evasive answer bias. *J. Amer. Statist. Assoc.* **60,** 63-69.

$f_1' = \left.\frac{\delta f(\bar{y}, u)}{\delta \bar{y}}\right]_{(\mu, 0)}$, $= 1$ second order partial derivative $\frac{\delta^2 f(\bar{y}, u)}{\delta \bar{y}^2} = 0$ and second order partial derivative $f_{12}'' = \left.\frac{\delta^2 f(\bar{y}, u)}{\delta \bar{y}\, \delta u}\right]_{(\mu, 0)} = \frac{f_2' \delta}{\mu}$ (with f_2' being the first order of $f(\bar{y}, u)$ with respect to u at the point $(\mu, 0)$ and δ taking one of the two values 'zero and unity' depending upon the particular form of an estimator. For example, for the estimator $\bar{y} + k u$, δ takes values zero whereas for the estimator $\bar{y}(1 - \mu)$, $\delta = 1$).

Some special cases of the generalized estimator t when σ^2 is unknown and k, g, α being the characterising scalars, are

(1) $t_1 = \bar{y} + k \frac{s^2}{n \bar{y}^2} = \bar{y} + k u$

(2) $t_2 = \bar{y} + k \frac{s^2}{n \bar{y}^2} \left(1 + g \frac{s^2}{n \bar{y}^2}\right)$

$= \bar{y} + k u (1 + g u)$

(3) $t_3 = \bar{y} \left[1 + \frac{k s^2}{n \bar{y}^2} \left(1 + g \frac{s^2}{n \bar{y}^2}\right)^{-\alpha}\right]$ by Singh [3]

$= \bar{y} \left|1 + k u (1 + g u)^{-\alpha}\right|$

(4) $t_4 = \bar{y} \left[1 - \frac{s^2}{n \bar{y}^2} \left(1 + \frac{s^2}{n \bar{y}^2}\right)^{-1}\right]$ by Srivastava [4, 5]

$= \bar{y} \left[1 - u (1 + u)^{-1}\right]$

(5) $t_5 = \bar{y} \left(1 - \frac{s^2}{n \bar{y}^2}\right)$ by Srivastava [4, 5]

$= \bar{y} (1 - u)$.

(6) $t_6 = \bar{y} \left[1 + \frac{k s^2}{n \bar{y}^2} \left(1 - \frac{k s^2}{n \bar{y}^2}\right)^{-1}\right]$ by Thompson [9]

$= \bar{y} \left[1 + k u (1 - k u)^{-1}\right]$

(7) $t_7 = \bar{y} \left(1 + g \frac{s^2}{n \bar{y}^2}\right)$ by Upadhyaya and Srivastava [10]

$= \bar{y} (1 + u)$

(8) $$t_8 = \bar{y}\left[1 + \frac{s^2}{n\bar{y}^2}\left(1 + \frac{s^2}{n\bar{y}^2}\right)^{-1}\right]$$ by Sahai and Ray [1]

$$= \bar{y}\left[1 + u(1 + u)^{-1}\right]$$

(9) $$t_9 = \bar{y}\left[1 + \frac{s^2}{n\bar{y}^2}\left(1 + \frac{s^2}{n\bar{y}^2}\right)^{-1}\right]$$ by Srivastava and Banarsi [6]

$$= \bar{y}\left[1 + u(1 + u)^{-2}\right]$$

(10) $$t_{10} = \bar{y}\left[1 + \frac{k s^2}{n\bar{y}^2}\left(1 + \frac{g s^2}{n\bar{y}^2}\right)^{-1}\right]$$ by Srivastava and Bhatnagar [7]

$$= [1 + k u(1 + g u)^{-1}]$$

(11) $$t_{11} = \bar{y}\left[1 + \frac{s^2}{n\bar{y}^2}\left(1 - \frac{s^2}{n\bar{y}^2}\right)^{-1}\right]$$ by Srivastava and Dwivedi [8]

$$= \bar{y}\left[1 + u(1 - u)^{-1}\right]$$

where various forms of the function $f(\bar{y}, u)$ are given by the expressions on right hand sides of (1) to (11) in terms of $\bar{y}$ and u.

It may be mentioned here that all the estimators listed from (1) to (11) belong to the class t and satisfy the condition $f(\mu, 0) = \mu$ with $f_1' = 1$ and $f_{12}^* = f_2'\delta/\mu, \delta = 1$ or 0.

2. *Bias and Mean square error of t*

To find the bias and mean square error (MSE) of t upto terms of order $0(n^{-2})$, let

$$\bar{y} = \mu + z \text{ and } s^2 = \sigma^2 + v \tag{2.1}$$

where z and v are of order $0(n^{-1/2})$ with $E(z) = E(v) = 0$, and $E(z)^2 = \frac{\sigma^2}{n} = \frac{\mu^2 C^2}{n}$.

With $\bar{y}^* = \mu + \theta(\bar{y} - \mu)$ and $u^* = \theta u, 0 < \theta < 1$, expanding $t = f(\bar{y}, u)$ in third order Taylor's series about the point $\mu, 0$, we have

$$t = f(\mu, 0) + (\bar{y} - \mu) \left.\frac{\delta f(\bar{y}, u)}{\delta \bar{y}}\right]_{(\mu, 0)} + u \left.\frac{\delta f(\bar{y}, u)}{\delta u}\right]_{(\mu, 0)}$$

$$+ \frac{1}{2!}\left\{ (\bar{y} - \mu)^2 \left.\frac{\delta f(\bar{y}, u)}{\delta \bar{y}^2}\right]_{(\mu, 0)} + 2(\bar{y} - \mu)\, u \left.\frac{\delta f(\bar{y}, u)}{\delta \bar{y}\, \delta u}\right]_{(\mu, 0)}\right.$$

$$\left. + u^2 \left.\frac{z\delta f(\bar{y}, u)}{\delta u^2}\right]_{(\mu, 0)} \right\} + \frac{1}{3!}\left\{\bar{y} - \mu) \frac{\delta}{\delta \bar{y}} + u \frac{\delta}{\delta u}\right\}^3 f(\bar{y}^* . u^*)$$

Now, we have $\left.\frac{\delta f(\bar{y}, u)}{\delta \bar{y}}\right]_{(\mu, 0)} = 1, \; \frac{\delta^2 f(\bar{y}, u)}{\delta \bar{y}^2} = 0,$

$\frac{\delta^3 f(\bar{y}, u)}{\delta \bar{y}^3} = 0, \frac{\delta^3 f(\bar{y}, u)}{\delta \bar{y}^2\, \delta u} = 0;$ and further for $\bar{y} - \mu = z$ and

$$\left[\frac{z}{\mu}\right] < 1,\; u = \frac{s^2}{n\,\bar{y}^2} = \frac{\delta^2 (1 + v/\sigma^2)}{n\mu^2 \left(1 + \frac{z}{\mu}\right)^2} = \frac{C^2}{n}\left(1 + \frac{V}{\sigma^2}\right)\left(1 + \frac{z}{\mu}\right)^{-2}$$

$$= \frac{C^2}{n}\left(1 + \frac{V}{\sigma^2}\right)\left(1 - \frac{2z}{\mu} + \ldots\right)$$

so that

$$t = \mu + z + \frac{C^2}{n}\left(1 + \frac{V}{\sigma^2}\right)\left(1 - \frac{2z}{\mu} + ..\right) f_2'$$

$$+ \frac{1}{2!}\left\{0 + 2z \frac{C^2}{n}\left(1 + \frac{V}{\sigma^2}\right)\left(1 + \frac{z}{\mu}\right)^{-2} f_{12}''\right.$$

$$\left. + \frac{C^4}{n^2}\left(1 + \frac{V}{\sigma^2}\right)^2 \left(1 + \frac{z}{\mu}\right)^{-4} f_2''\right\} + \frac{1}{3!}\left\{0 + 3(\bar{y} - \mu)^2\right.$$

$$. u \frac{\delta^2 f(\bar{y}^*, u^*)}{\delta \bar{y}^{*2}\, \delta u^*} + 3(\bar{y} - \mu) \cdot u^2 \frac{\delta^3 f(\bar{y}^*, u^*)}{\delta \bar{y}^*\, \delta u^{*2}}$$

$$\left. + u^3 \frac{\delta^3 f(\bar{y}^*, u^*)}{\delta u^{*3}}\right\}$$

$$= \mu + z + \frac{C^2}{n}\left(1 - \frac{V}{\sigma^2}\right)\left(1 - \frac{2z}{\mu} + 3\frac{z^2}{\mu^2} - 4\frac{z^3}{\mu^3} + 5\frac{z^5}{\mu^5} - \dots\right) f_2'$$

$$+ \frac{1}{2!}\left\{\frac{2z\,C^2}{n}\left(1 + \frac{v}{\sigma^2}\right)\left(1 - \frac{2z}{\mu} + \frac{3z^2}{\mu^2} - \frac{4z^3}{\mu^3} + \frac{5z^4}{\mu^4} - \dots\right) f_{12}''\right.$$

$$\left. + \frac{C^4}{n^2}\left(1 + \frac{v}{\sigma^2}\right)^2 \left(1 + \frac{z}{\mu}\right)^{-4} f_2''\right\} + \frac{1}{3!}\left\{0 + 3(\bar{y} - \mu)^2 \frac{u\,\sigma^2 f(y^*, u^*)}{\delta\,\bar{y}^{*2}\,\delta u^*}\right.$$

$$\left. + 3(\bar{y} - \mu)\,u^2\,\delta^3 \frac{f(\bar{y}^*, u^*)}{\delta\,\bar{y}^*\,\delta u^{*2}} + u^3\,\delta^3 \frac{f(\bar{y}^*, u^*)}{\delta u^{*3}}\right\}$$

$$= \mu + z + \frac{1}{n}\left\{\left(C^2 - 2C^2\frac{z}{\mu} + \frac{V}{\mu^2} + 3C^2\frac{z^2}{\mu^2} - \frac{2zV}{\mu^3}\right) - 4C^2\frac{z^3}{\mu^3}\right.$$

$$\left. + 3\frac{z^2 V}{\mu^4} + 5C^2\frac{z^4}{\mu^4} - 4\frac{z^2 V}{\mu^6} + \dots\right\} f_2' + \frac{1}{n}\left\{\left(C^2 z - 2C^2\frac{z^2}{\mu} + \frac{z\,V}{\mu^2}\right)\right.$$

$$+ 3C^2\frac{z^3}{\mu^2} - 2\frac{z^2 V}{\mu^3} - 4C^2\frac{z^4}{\mu^3}$$

$$\left. + 3\frac{z^2 V}{\mu^4} + 5C^2\frac{z^5}{\mu^4} - 4\frac{z^4 V}{\mu^5} + 5\frac{z^5 V}{\mu^6} + \dots\right\} f_{12}''$$

$$+ \frac{C^4}{2n^2}\left(1 + \frac{V}{\sigma^2}\right)^2\left(1 + \frac{z}{\mu}\right)^{-4} f_2'' + \frac{1}{3!}\left\{3(\bar{y} - \mu)^2 u\,\delta^3 \frac{f(\bar{y}^*, u^*)}{\delta\,\bar{y}^{*2}\,\delta u^*}\right.$$

$$\left. + 3(\bar{y} - \mu)\,u^2 \frac{\delta^3 f(\bar{y}^*, u^*)}{\delta\,\bar{y}^*\,\delta u^{*2}} + u^3 \frac{\delta^3 f(\bar{y}^*, u^*)}{\delta \mu^{*3}}\right\} \qquad (2.3)$$

Taking expectation in (2.3), to the terms of order $0(n^{-2})$, for symmetrical populations, we have

$$E(t) = \mu + E\frac{C^2}{n}\left(1 + \frac{3z^2}{\mu^2}\right) f_2' - \frac{2z^2 c^2}{n\,\mu} f_{12}'' + \frac{C^4}{2n^2} f_2''$$

$$= \mu + \frac{C^2}{n}\left(1 + \frac{3C^2}{n}\right) f_2' - \frac{2\mu\,C^4}{n} f_{12}'' + \frac{C^4}{2n^2} f_2''$$

or Bias $(t) = E(t) - \mu$

$$= \frac{C^2}{n}\left[\left(1 + \frac{C^2}{n}\right) f_2' + \frac{2C^2}{n}\left(f_2' - \mu\,f_{12}''\right) + \frac{C^2}{2n} f_2''\right] \qquad (2.4)$$

Again, from (2.3), we have

$$\text{MSE}(t) = E(t-\mu)^2$$

$$= E\Bigg[z + \frac{1}{n}\Bigg\{\left(C^2 - 2C^2\frac{z}{\mu} + \frac{V}{\mu^2} + 3C^2\frac{z^2}{\mu^2} - \frac{2zV}{\mu^3}\right) - 4C^2\frac{z^3}{\mu^3}$$

$$+ 3\frac{z^2V}{\mu^4} + 5C^2\frac{z^4}{\mu^4} - 4\frac{z^3V}{\mu^5} + 5\frac{z^4V}{\mu^6} + \ldots\Bigg\} f_2'$$

$$+ \frac{1}{n}\Bigg\{\left(C^2 z - 2C^2\frac{z^2}{\mu} + \frac{zV}{\mu^2}\right) + 3C^2\frac{z^3}{\mu^2} - \frac{2z^2V}{\mu^3} - 4C^2\frac{z^4}{\mu^3}$$

$$+ 3\frac{z^3V}{\mu^4} + 5C^2\frac{z^5}{\mu^4} - \frac{4z^4V}{\mu^5} + \frac{5z^5V}{\mu^6} + \ldots\Bigg\} f_{12}''$$

$$+ \frac{C^4}{2n^2}\left(1 + \frac{V}{\sigma^2}\right)^2\left(1 + \frac{z}{\mu}\right)^{-4} f_2''$$

$$+ \frac{1}{3!}\Bigg\{3(\bar{y}-\mu)^2 u \frac{\delta^3 f(\bar{y}^*, u^*)}{\delta \bar{y}^{*2}\,\delta u^*} + 3(\bar{y}-\mu)u^2\frac{\delta^3 f(\bar{y}^*, u^*)}{\delta \bar{y}^*\,\delta u^{*2}}$$

$$+ u^3\,\frac{\delta^3 f(\bar{y}^*, u^*)}{\delta u^{*3}}\Bigg\}\Bigg]^2$$

whence, upto terms of order $0(n^{-2})$, the mean square error of t is

$$\text{MSE}(t) = E\left[z^2 + \frac{C^4}{n^2}(f_2')^2 + \frac{2z}{n}\left(C^2 - 2C^2\frac{z}{\mu} + \frac{V}{\mu^2}\right)f_2' + 2\frac{z^2}{n}C^2 f_{12}''\right]$$

from which, for symmetrical populations, upto terms of order $0(n^{-2})$, the mean square of t is

$$\text{MSE}(t) = \frac{\mu^2C^2}{n} + \frac{C^4}{n^2}(f_2')^2 - 4\mu\frac{C^4}{n^2}f_2' + \frac{2\mu^2C^4}{n^2}f_{12}''$$

$$= \frac{\mu^2C^2}{n}\left[1 + \frac{C^2}{n}\left\{\frac{(f_2')^2}{\mu^2} - \frac{4f_2'}{\mu} + \frac{2f_2'\delta}{\mu}\right\}\right] \quad (2.5)$$

which is minimised for

$$f_2' = \mu(2-\delta) \quad (2.6)$$

where δ takes one of the two values '0 and 1' and the minimum mean square error is given by

$$\text{MSE}(t)_{\text{min.}} = \frac{\mu^2 C^2}{n}\left[1 + \frac{C^2}{n}\left\{(2-\delta)^2 - 4(2-\delta) + 2(2-\delta)\delta\right\}\right]$$

$$= \frac{\mu^2 C^2}{n}\left[1 - \frac{C^2}{n}(2-\delta)^2\right] \tag{2.7}$$

3. *Concluding Remarks*

(a) From (2.6) and (2.7), the class of estimators t attains its minimum value for $f_2' = \mu(2-\delta)$, $\delta = \mu f_{12}''/f_2'$, and the minimum mean square error is

$$\text{MSE}(t)_{\text{min}} = \frac{\mu^2 C^2}{n}\left[1 - \frac{C^2}{n}(2-\delta)^2\right] \tag{3.1}$$

Thus, any estimator from the class t cannot have mean square error less than the expression given by (3.1).

(b) Bias, mean square error and the related results to the estimators listed in section 1 may easily be found as special cases of this study. For example, with k, g and α being the characterizing scalars for the estimator.

$$t_3 = \bar{y}\left[1 + \frac{k s^2}{n \bar{y}^2}\left(1 + \frac{g s^2}{n \bar{y}^2}\right)^{-\alpha}\right]$$

$$= \bar{y}\left[1 + k u (1 + g u)^{-\alpha}\right]$$

by Singh [3], we have $f_{12}'' = k$, $f_2' = k\mu$, $\delta = \dfrac{\mu f_{12}''}{f_2'} = 1$ so that $f_2' = \mu(2-\delta) = k\mu$ satisfying (2.6) gives the value of $k = 1$ for which MSE (t_3) is minimised and the minimum mean square error

$$\text{MSE}(t_3)_{\text{min.}} = \frac{\mu^2 C^2}{n}\left(1 - \frac{C^2}{n}\right) \tag{3.2}$$

is obtained from (3.1) by putting $\delta = 1$. Further, for the estimator t_3, we have $f_2' = k\mu$, $f_{12}'' = k$ and $f_2'' = -2\alpha k g \mu$, so that the bias of of t_3 from (2.4) is

$$\text{Bias}(t_3) = \frac{k\mu C^2}{n}\left[1 + \frac{C^2}{n}(1 - \alpha g)\right] \tag{3.3}$$

which, for $k = g = 1$, reduces to

$$\text{Bias}(t_3) = \frac{\mu C^2}{n}\left[1 - (\alpha - 1)\frac{C^2}{n}\right]. \tag{3.4}$$

It may be mentioned here that the expressions (3.2) and (3.4) are the same expressions as obtained by Singh [3]. Similarly, the results of all the estimators listed in section 1 may easily be shown to be special cases of those of the generalized estimator t.

(c) For the estimators having $f_{12}'' = 0$, that is $\delta = 0$, we have from (2.7)

$$MSE(t)_{min.} = \frac{\mu^2 C^2}{n}\left(1 - \frac{4C^2}{n}\right) \tag{3.5}$$

For example, the estimator $t_1 = \bar{y} + k\frac{s^2}{n\bar{y}^2} = \bar{y} + k\,u$, k being the characterizing scalar, has $f_2' = k$, $f_{12}'' = 0$ and $\delta = 0$ so that it attains, for the optimum value $f_2' = \mu(2 - \delta) = k$ satisfying (2.6) and giving $k = 2\mu$, the minimum mean square error given by (3.5).

(d) The estimator like t_3 has the practical advantage over the estimator like t_1, since the optimum value $k = 1$ minimizing mean square error for t_3 is independent of parameter whereas the optimum value $k = 2\mu$ in case of t_1 depends upon the parameter μ. In fact, for the sub-set of estimators of the form $t_s = \bar{y}(u)$ of the class t where $h(u)$ is the function of u such that $h(0) = 1$. there is no practical difficulty in using the optimum value $f_2' = \mu h'(0) = \mu$ (the value of δ for t_s is unity and $h'(0)$ is the first derivative of $h(u)$ with respect to u at $u = 0$) giving $h'(0) = 1$, a quantity independent of the parameter.

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REFERENCES

[1] Sahai, A. and Ray, S.K., 1979. On estimating the mean of symmetrical populations. *Jour. Ind. Soc. Ag. Statistics*, **31** (3), 45-48.

[2] Searles, D.T., 1964. The utilization of a known coefficient of variation in estimation procedure. *Jour. Amer. Statist. Assoc.*, **59**, 1225-26.

[3] Singh, H.P., 1987. A note on estimating the mean of symmetrical populations. *Jour. Ind. Soc. Ag. Statistics*, **39** (2), 171-180.

[4] Srivastava, V.K., 1974. On the use of the coefficient of variation in estimating mean. *Jour. Ind. Soc. Ag. Statistics.*, **26** (2), 33-36.

[5] Srivastava, V.K., 1980. A note on the estimation of mean in normal population, *Metrika*, **27**, 99-102.

(6) Srivastava, S.R. and Banarsi, 1982. A note on the estimation of mean of a symmetrical population. *Jour. Ind. Soc. Ag. Statistics*, **34** (3). 79-81.

[7] Srivastava, V.K. and Bhatnagar, S. 1983. A revisit to the use of coefficient of variation in estimating mean *Jour. Ind. Soc. Ag. Statistics*, **35** (1), 67-71.

[8] Srivastava, V.K. and Dwivedi, T.D., 1985. Improving estimation of population mean : Some remarks, *Jour. Ind. Soc. Ag. Statistics*, **37** (2), 154-157.

[9] Thompson, J.R., 1986. Some shrinkage techniques for estimating the mean. *Jour, Amer. Statist. Assoc.*, **63**, 113-23.

[10] Upadhyaya, L.N. and Srivastava, S.R., 1976. A note on the use of coefficient of variation in estimating mean. *Jour. Ind. Soc. Ag. Statistics*, **28** (2), 97-99.