APPLICATIONS OF SOME GENERALISATIONS OF KRONECKER PRODUCT IN THE CONSTRUCTION OF FACTORIAL DESIGNS

RAHUL MUKERJEE

Department of Statistics University of Calcutta

(Received : February, 1982)

SUMMARY

Some generalisations of Kronecker product are proposed for the construction of effectwise orthogonal factorial designs. The methods suggested have wide applicability, ensure desirable properties with respect to main effects, lead to flexibility in block size and generally require a small number of replicates.

INTRODUCTION

A factorial design is called *effectwise orthogonal* if the best linear estimates of estimable treatment contrasts belonging to different factorial effects are orthogonal (*i.e.* uncorrelated) so that the adjusted treatment sum of squares can be split up orthogonally into components due to different factorial effects which can be shown in the same analysis of variance table.

The problem of construction of effectwise orthogonal factorial designs starting from some particular sufficient conditions was considered by John [2], Dean and John [1] etc. Recently Mukerjee [5] proposed some broader methods starting from a necessary and sufficient condition for effectwise orthogonality. In the present work following the line of Mukerjee [5] some further methods of construction are proposed utilising some generalisations of Kronecker product. The methods suggested have wide applicability, ensure desirable properties with respect to main effects, lead to flexibility in block size and generally require a small number of replicates.

NOTATIONS AND PRELIMINARIES

Essentially we follow the system of notations introduced in Mukerjee (1981). Consider a factorial experiment involving m factors

$F_1,\ldots,F_m$, the j th factor being at s_j ($\geqslant 2$) levels, $1 \leqslant j \leqslant m$. Let the $v \left(= \pi_{j=1}^{m} s_j\right)$ level combinations be arranged in a block design with b blocks and incidence matrix $N^{(v \times b)}$. The fixed effects intrablock model with no block-treatment interaction and with a constant error variance is assumed. Throughout this paper the v level combinations will be lexicographically ordered (cf. Kurkjian and Zelen [3]).

Definition 2.1. A proper matrix is a square matrix with all row sums and column sums equal.

For an equireplicate factorial experiment in a block design with constant block size and with incidence matrix N, the following theorem was proved by Mukerjee [4] :

Theorem 2.1. A necessary and sufficient condition for the design to be effectwise orthogonal is that NN' is of the form

$$NN' = \sum_{g=1}^{w} \xi_g \left(\underset{j=1}{\overset{m}{\times}} V_{gj}\right), \qquad \ldots(2.1)$$

where $\times$ is Kronecker product, $\underset{j=1}{\overset{m}{X}} V_{gj} = V_{g1} \times V_{g2} \times \ldots \times V_{gm}$, w is a positive integer, $\xi_1,\ldots\xi_w$ are some real numbers and for each g, V_{gj} is some proper matrix of order s_j.

If NN' satisfies the condition stated above, following Mukerjee (1979) it is said to have *structure K*.

Because of Theorem 2.1, in the construction of effectwise orthogonal designs it seems natural to start with some form of product of simpler designs. As pointed out by Mukerjee [5], the ordinary Kronecker product does not serve our purpose since it may make the block size and/or the number of replicates of the ultimate design too large. Therefore one has to consider some generalisations of Kronecker product. Some such generalisations have been considered in Mukerjee [5]. A few more are going to be presented here.

Generalised Cyclic Product

As in Mukerjee (1981), the *union* of p block designs $N_1^*,\ldots,N_p^*$ involving the same set of treatments is a design with incidence matrix

$$\bigcup_{l=1}^{p} N_l^* = (N_1^*,\ldots,N_p^*). \qquad \ldots(3.1)$$

when the initial block designs are labelled by two or more subscripts they are arranged in lexicographic order in the right-hand member of (3.1).

For $1 \leqslant j \leqslant m$, let D_j be a varietal design (not necessarily binary) in b_j blocks, s_j varieties (denoted by $0, 1, \ldots, s_j - 1$) with common replication number r_j, constant block size k_j and incidence matrix $N_j^{(s_j \times b_j)}$). Let for $1 \leqslant j \leqslant m$,

$$N_j^{(s_j \times b_j)} = \sum_{l=0}^{u-1} N_{jl}^{(s_j \times b_j)}, \qquad \ldots(3.2)$$

where u is a positive integer and elements of N_{jl} are nonnegative integers.

Definition 3.1. The generalised cyclic product of order $t(1 \leqslant t \leqslant m)$ of $N_1, \ldots, N_m$ with respect to the decomposition (3.2) is a design with incidence matrix

$$N^{(1)} = \bigcup_{h_{t+1}, \ldots h_m = 0}^{u-1} \Big[\sum_{l_1, \ldots l_t = 0}^{u-1} \big(\mathop{X}_{j=1}^{t} N_{jl_j} \big)$$

$$\times \big(\mathop{X}_{j=t+1}^{m} N_{j, l_1 + \cdots + l_t + h_j} \big) \Big], \qquad \ldots(3.3)$$

for $t+1 \leqslant j \leqslant m$, $l_1 + \ldots + l_t + h_j$ being reduced mod u.

If $t=1$, generalised cyclic product of order t reduce to cyclic product introduced in Mukerjee [5] and if $t=m$ or $u=1$ it reduces to ordinary Kronecker product.

Associating the rows of $N^{(1)}$ with the v level combinations following lexicographic order, for the resulting m-factor design the following theorem can be proved.

Theorem 3.1 If for each j, $l (1 \leqslant j \leqslant m : 0 \leqslant l \leqslant u-1)$ the design N_{jl} has constant block size $u^{-1} k_j$, then the m-factor design $N^{(1)}$ is effectwise orthogonal.

Proof. By (3.1), (3.3) and the standard rules for operation with partitioned matrices and Kronecker products

$$N^{(1)} N^{(1)\prime} = \sum_{h_{t+1}, \ldots, h_m = 0}^{u-1} \quad \sum_{l_1, \ldots, l_t = 0}^{u-1}$$

$$\sum_{a_1,\ldots,a_t=0}^{u-1} [\overset{t}{\underset{j=1}{X}} (N_{jl_j} N'_{ja_j})]$$

$$\times [\overset{m}{\underset{j=t+1}{X}} (N_{j,l_1+\ldots+l_t+h_j} N'_{j,a_1+\ldots+a_t+h_j})]$$

$$=\sum_{h_{t+1},\ldots,h_m=0}^{u-1} \sum_{l_1,\ldots l_t=0}^{u-1} \sum_{\beta_1,\cdots,\beta_t=0}^{u-1} [\overset{t}{\underset{j=1}{X}} (N_{jl_j} N'_{j,l_j+\beta_j})]$$

$$\times (\overset{m}{\underset{j=t+1}{X}} (N_{j,l_1+\cdots+l_t+h_j} N'_{j,l_1+\cdots+l_t+\beta_1+\cdots+\beta_t+h_j})]$$

$$=\sum_{\beta_1,\cdots,\beta_t=0}^{u-1} [\overset{t}{\underset{j=1}{X}} (\sum_{l_{jm}=0}^{u-1} N_{jl_j} N'_{j,l_j+\beta_j})]$$

$$[\overset{m}{\underset{j=t+1}{X}} (\sum_{l_j=0}^{u-1} N_{jl_j} N'_{j,l_j+\beta_1+\ldots+\beta_t})], \qquad \ldots(3.4)$$

where, $l_j+\beta_j, l_j+\beta_1+\ldots+\beta_t$ etc are reduced mod u. Since for $1 \leqslant j \leqslant m$, N_j are equireplicate with common replication number r_j, it is easy to check that under the condition of the theorem for each

$$\beta_1,\ldots,\beta_t, \sum_{l_j=0}^{u-1} N_{jl_j} N'_{j,l_j+\beta_j} \quad (1 \leqslant j \leqslant t)$$

and

$$\sum_{l_j=0}^{u-1} N_{jl_j} N'_{j,l_j+\beta_1+\ldots+\beta_t} \quad (t+1 \leqslant j \leqslant m) \text{ are proper}$$

matrices with each row sum and column sum $u^{-1} r_j k_j$. Hence by (3.4), $N^{(1)} N^{(1)'}$ has structure K. Further the design $N^{(1)}$ has constant block size $u^{-(m-t)} \prod_{j=1}^{m} k_j$ and has common replication number $\prod_{j=1}^{m} r_j$. Hence the result follows by Theorem 2.1. *Q.E.D.*

Following Mukerjee [5] a simple procedure is described for getting the matrices $N_{jl} (1 \leqslant j \leqslant m; 0 \leqslant l \leqslant u-1)$ such that the conditions of Theorem 3.1 are satisfied. For $1 \leqslant j \leqslant m$, denote the varieties in D_j

by $0,1,\ldots,s_j-1$ and let $Z_j^{(k_j \times b_j)}$ be an array formed by writing the blocks of D_j as columns. Suppose for each j, k_j is an integral multiple of u.

Then partitioning Z_j' into u subarrays each with $u^{-1}k_j$ coloumns as

$$Z_j' = (Z_{j0}', \ldots, Z_{j,u-1}')$$

in order statisfy the conditions of Theorem 4.1, it is enough to take for each l $(0 \leqslant l \leqslant u-1)$ the matrix N_{jl} as the incidence matrix of a varietal design with blocks given by the columns of Z_{jl}.

Thus the method of generalised cyclic product is widely applicable. The following example illustrates the method.

Example 3.1. To construct a $2 \times 3 \times 5$ design let D_j $(j=1,2,3)$ be such that

$$Z_1 = \begin{matrix} 0 \\ 1 \end{matrix}, \quad Z_2 = \begin{matrix} 0\ 1\ 2 \\ 1\ 2\ 0 \end{matrix}, \quad Z_3 = \begin{matrix} 1\ 2\ 3\ 4\ 0 \\ 4\ 0\ 1\ 2\ 3 \end{matrix}.$$

Here $k_j=2$ $(j=1,2,3)$, $r_1=1, r_2=r_3=2$. For $u=2$, k_1,k_2,k_3 are integral multiples of u. Hence taking $u=2$, we may form matrices N_{jl} as stated earlier from the subarrays Z_{jl} $(j=1, 2, 3;\ l=0, 1)$. Thus.

$$N_{10} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad N_{11} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad N_{20} = \begin{bmatrix} 100 \\ 010 \\ 001 \end{bmatrix}, \quad N_{21} = \begin{bmatrix} 001 \\ 100 \\ 010 \end{bmatrix},$$

$$N_{30} = \begin{bmatrix} 00001 \\ 10000 \\ 01000 \\ 00100 \\ 00010 \end{bmatrix}, \quad N_{31} = \begin{bmatrix} 01000 \\ 00100 \\ 00010 \\ 00001 \\ 10000 \end{bmatrix}.$$

Now using generalised cyclic product of order 2 (as in (3.3)) it is possible to construct a $2 \times 3 \times 5$ design with incidence matrix

$$\Big[\Big(\sum_{l_1,l_2=0}^{1} N_{1l_1} \times N_{2l_2} \times N_{3,l_1+l_2}, \quad \sum_{l_1,l_2=0}^{1} N_{1l_1} \times N_{2l_2} \times N_{3,l_1+l_2+1} \Big), \Big]$$

where l_1+l_2, l_1+l_2+1 are reduced mod 2. The above design requires four replicates and involves blocks of size 4. By Theorem 3.1, the design is effectwise orthogonal. Writing the design in full it can also seen to be connected.

Some further properties of generalised cyclic product and their illustration with reference to the above example will be considered in the last section.

KHATRI RAO PRODUCT

In this section we shall use a special product, introduced by Khatri and Rao in a different context. We shall use the product in form given in Rao (1973, pp 30).

For $1 \leqslant j \leqslant m$, let D_j, N_j be as in the preceding section. Let for $1 \leqslant j \leqslant m$, Nj be partitioned as

$$N_j=(N_{j0},N_{j1},\cdots,N_{j,u-1}), \qquad \ldots(4.1)$$

where u is a positive integer and for each l $(0 \leqslant l \leqslant u-1)$, N_{jl} represents a varietal design in s_j varieties with constant block size k_j.

Definition 4.1. The Khatri-Rao product of $N_1,\ldots,N_m$ with respect to the partition (4.1) is a design with incidence matrix

$$N^{(2)}=(\underset{j=1}{\overset{m}{X}} N_{j0}, \underset{j=1}{\overset{m}{X}} N_{j1},\ldots, \underset{j=1}{\overset{m}{X}} N_{j,u-1}) \qquad \ldots(4.2)$$

(of Rao (1973, pp 30)).

The Khatri-Rao product reduces to ordinary Kronecker product if $u=1$. Associating the rows of $N^{(2)}$ with v level combinations following lexicographic order, for the resulting $m-$factor design the following theorem can be proved :

Theorem 4.1. If for each $j,l (1 \leqslant j \leqslant m;\ 0 \leqslant l \leqslant u-1)$, the design N_{jl} be equireplicate (with common replication number say, r_{jl}), then the m$-$factor design $N^{(2)}$ is effectwise orthogonal.

Proof. Under the given condition, it readily follows that the matrix $N_{jl}N'_{jl}$ is proper (with each row sum and column sum $k_j r_{jl}$) for each j,l $(1 \leqslant j \leqslant m;\ 0 \leqslant l \leqslant u-1)$. Hence by (4.2),

$$N^{(2)}N^{(2)\prime}=\sum_{l=0}^{u-1} (\underset{j=1}{\overset{m}{X}} N_{jl}N'_{jl})$$

has structure K. Further, it is easy to check that the design $N^{(2)}$ has constant block size

$$\prod_{j=1}^{m} k_j$$

and is equireplicate with common replication number

$$\sum_{l=0}^{u-1} \prod_{j=1}^{m} (\pi_{r_{jl}}).$$

Hence the result follows by Theorem 2.1. *Q.E.D.*

We now describe a simple procedure for getting N_{jl} ($1 \leqslant j \leqslant m$; $0 \leqslant l \leqslant u-1$) such that the conditions of the above theorem are satisfied. For $1 \leqslant j \leqslant m$, defining the array Z_j corresponding to D_j as in the preceding section, suppose it is possible to partition Z_j into subarrays

$$Z_j = (Z_j^{(0)}, Z_j^{(1)}, \ldots, Z_j^{(u-1)}) \quad \ldots (4.3)$$

such that in $Z_j^{(l)}$ $(0 \leqslant l \leqslant (u-1))$ the symbols $0, 1, \ldots, s_{j-1}$ occur equal number of times. For each j, l, taking N_{jl} as the incidence matrix of the varietal design with blocks given by the columns of $Z_j^{(l)}$, the conditions of Theorem 4.1 are satisfied.

The following example illustrates the method.

Example 4.1. To construct a 5×9 design let $D_j (j=1, 2)$ be such that

$$Z_1 = \begin{matrix} 0 & 2 & 4 & 1 & 3 & 0 & 4 & 3 & 2 & 1 \\ 1 & 3 & 0 & 2 & 4 & 2 & 1 & 0 & 4 & 3 \end{matrix}, \quad Z_2 = \begin{matrix} 0 & 1 & 2 & 0 & 3 & 6 \\ 3 & 4 & 5 & 1 & 4 & 7 \\ 6 & 7 & 8 & 2 & 5 & 8 \end{matrix}.$$

We partition Z_1, Z_2 as in ...(4.3)

with $u=2$ and

$$Z_1^{(0)} = \begin{matrix} 0 & 2 & 4 & 1 & 3 \\ 1 & 3 & 0 & 2 & 4 \end{matrix}, \quad Z_1^{(1)} = \begin{matrix} 0 & 4 & 3 & 2 & 1 \\ 2 & 1 & 0 & 4 & 3 \end{matrix},$$

$$Z_2^{(0)} = \begin{matrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{matrix}, \quad Z_2^{(1)} = \begin{matrix} 0 & 3 & 6 \\ 1 & 4 & 7 \\ 2 & 5 & 8 \end{matrix}.$$

Then the conditions of Theorem 4.1 hold with $k_1=2$, $k_2=3$, $r_{10}=r_{11}=2$, $r_{20}=r_{21}=1$. Now taking the matrices N_{jl} ($j=1, 2$; $l=0, 1$) as stated earlier and applying Khatri-Rao product a 5×9 design in blocks of size 6 in only four replications can be constructed. By Theorem 4.1, the design is effectwise orthogonal. Writing the design in full it can also be seen to be connected.

Some further properties of Khatri-Rao product and their illustration with reference to the above example will be considered in the next section.

Further Properties of the Methods

We have, in this paper, suggested two distinct methods for the construction of effectwise orthogonal designs starting from equireplicate varietal designs $D_j(1 \leqslant j \leqslant m)$. Following the line of Mukerjee [5], it can be shown under quite general conditions these methods are faithful with regard to main effects in the sense of Mukerjee [5]. In other words, under quite general conditions these methods transmit the properties (in terms of loss of information on different contrasts) of the varietal design D_j to main effect F_j in the ultimate factorial design $(1 \leqslant j \leqslant m)$. Denoting the ultimate factorial design by D, this means in particular that if the method of construction be faithful then (*i*) all contrasts belonging to main effect F_j are estimable in D if D_j be connected, (*ii*) main effect F_j is balanced in D if D_j be balanced, (*iii*) main effect F_j is partially balanced in D if D_j be partially balanced, (*iv*) full information is retained on main effect F_j in D if full information be retained on all varietal contrasts in D_j. Thus the properties of D with respect to the main effects can be controlled by suitably choosing the varietal designs D_j, $1 \leqslant j \leqslant m$.

In particular, the method of generalised cyclic product is faithful under the conditions of Theorem 3.1. As a consequence, noting that in Example 3.1, D_1 retains full information on all varietal contrasts, D_2 is Balanced, D_3 is partially balanced it follows that in the corresponding $2 \times 3 \times 5$ design full information is retained on main effect F_1, main effect F_2 is balanced and main effect F_3 is partially balanced.

Following the line of Mukerjee [5] it can also be shown that the method of Khatri-Rao product is faithful under the conditions of Theorem 4.1 if further $r_{j1} = u^{-1} r_j$ for each j, 1(*i.e.* if further for each j, the design D_j be $u^{-1} r_j$-resolvable). It is readily seen that these conditions are satisfied by D_1, D_2 considered in Example 4.1. Since in that example D_1 is balanced and D_2 is partially balanced it follows that in the corresponding 5×9 design main effect F_1 is balanced and main effect F_2 is partially balanced.

As noted earlier, the methods described generally require a small number of replicates and are flexible with regard to block size. In this connexion, attention may be drawn to the $2\times3\times5$ design proposed in Example 3.1. The design involves blocks of size 4 and it may be noted that 4 has no common multiple with 3 or 5. Also by suitably choosing D_j $(1\leqslant j\leqslant m)$ it is often possible to ensure connectedness of the utlimate factorial designs.

The properties of the factorial designs constructed by the methods described here with respect to interactions may be explored using the formulae on average loss of information presented in Mukerjee [5]. Also for the expressions for sum of squares due to different factorial effects, refer to Mukerjee [5].

The methods presented here together with those described in Mukerjee [5] have a very wide coverage. In fact for any given m (number of factors) and any given $s_1,\ldots,s_m$ (number of levels of different factors), applying these methods it is possible to generate a wide variety of effectwise orthogonal designs controlling the properties with respect to main effects suitably. In any particular situation, the experimenter should take into account practical considerations regarding block size and number of replicates, investigate the properties of the available designs with respect to interactions and apply his discretion to make a choice from amongst the available designs.

ACKNOWLEDGEMENT

The author is grateful to Professor S.K. Chatterjee, Department of Statistics, University of Calcutta, for his kind supervision of the work.

REFERENCES

[1] Dean, A.M. and John, J.A. (1975) : Single replicate factorial experiments in generalised cyclic designs : II. Assymetrical arrangements. *J. Roy. Statist. Soc. Ser.* **37**, 72-76.

[2] John, J.A. (1973) : Generalised cyclic designs in factorial experiments. *Biometrika*. **60**, 55-63.

[3] Kurkjian, B. and Zelen. M. (1963) Applications of the calculus of factorial arrangements I. Block and direct product designs. *Biometrika*, **50**, 63-73.

[4] Mukerjee, R. (1979) : Inter-effect orthogonality in factorial experiments. *Calcutta Statist. Assoc. Bull.* 28, 83-108.

[5] Mukerjee, R. (1981) : Construction of effectwise orthogonal factorial designs. *J. Statist. Planning Infer*, **5**, 221-229.