GENERATING FUNCTION OF MODIFIED BRANCHING PROCESS

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1. INTRODUCTION

In an earlier paper [1] on branching process a modified model of the process was introduced wherein the probability of fission of a unit into sub-units was assumed to depend on the generation size. The probability of extinction of the population was deduced only under some assumptions about the constants in the process. In this paper, first the recurrence relation satisfied by the generating function of the process starting with a single unit and with i units are given. The probability of extinction is also obtained and its monotonocity is discussed for all non-negative values of α and β. Further the limiting form of the generating function as the number of generations becomes infinitely large is derived and for $\beta > 0$ the process is shown to be positive recurrent.

2. RECURRENCE RELATION

Markovian sequence of non-negative integer valued random variable $Zi(i=0, 1, 2, \ldots\ldots)$ such that $P(Z_1=r \mid z_0=1)=P_1(r)$ and $P_{10}+P_{11}<1$ is a modified branching process if the probability $\phi(j, r)$ that any unit in the population of size j generates r individuals as a decreasing function of j, the population size. All the probabilities are independent. The variate Z_n represents the number of the units in the nth generation.

Let $\quad \phi(j, r)=e^{-m_j}\dfrac{m_j^r}{r!}$

$$m_j=\alpha+\frac{\beta}{j}\,;\ \alpha+\beta=ni,$$

and constants α and β are such that

$$(i)\ \alpha>0,\ \beta>0,\ (ii)\ \alpha=0,\ \beta>0 \text{ and } (iii)\ \alpha>0,\ \beta=0.$$

Defining the generating functions for the first and nth generation by

$$f_1(s)=\sum_{j=0}^{\infty} P_{1j}s^j \quad \text{and} \quad f_n(s)=\sum_{j=0}^{\infty} P_{ij}^{n}\, s^j$$

where $|s|<1$ and P_{ij}^{n} is the probability of there being j units in the nth generation with i initial units.

Then,

$$\underset{n+1}{f(s)} = \sum P_{ij}^{n} \left\{ \sum_{r=0}^{\infty} \frac{e^{-(\alpha+\beta/j)}\,(\alpha+\beta/j)^r s^r}{r\,!} \right\}$$

$$=e^{\beta(s-1)}\, f_n(e^{\alpha(s-1)})=e^{\beta(s-1)}\, f_n s_1(s))$$

$$=e^{\beta[(s-n)+s_1(s)+s_2(s)+..\, s_{n-1}(s)]} f_1(s_n(s))$$

$$=e^{\beta S_{n-1}}\, f_1(s_n(s)) \qquad \ldots(1)$$

where

$$s_n(s)=e^{\alpha[s_{n-1}(s)-1]},$$

$$S_{n-1}(s)=\sum_{i=0}^{n-1} [s_i(s)-1] \text{ and } s_0(s)=s$$

For i initial units this will be

(*i*) $f_{n+1}(s)=e^{\beta S_{n-1}(s)}\,.\,e^{(\alpha i+\beta)(S_n(s)-1)}$ for $\alpha>0$, $\beta>0$

(*ii*) $f_{n+1}(s)=e^{\alpha i(s_n(s)-1)}$ for $\beta=0$, $\alpha>0$

(*iii*) $f_{n+1}(s)=e^{\beta(s-1)}$ $\qquad \alpha=0$

Substituting $s=0$, (1) gives the probabilities of extinction in the nth generation. Thus

(*i*) $f_{n+1}(0)=e^{\beta[S_{n-1}-1]} \quad f_1(s_n)$

where $s_n(0)=s_n$ and $S_{n-1}=\sum_{i=1}^{n-1} (s_i-1)$

(*ii*) $f_{n+1}(0)=f_1(s_n)$ for $\beta=0$

(*iii*) $f_{n+1}(0)=e^{-\beta}$ for $\alpha=0$

3. Probability of Extinction

It is shown in [1] that s_n is a monotonically increasing sequence and bounded by 1. Since s_{n-1} is a sum of negative terms (s_i-1) we consider the corresponding sum S_{n-1}^* of positive terms $(1-s_i)$. The sum

$$S_{n-1}^*=\sum_{i=0}^{\infty}(1-s_i)$$

is monotonically increasing. For $\alpha=1$, the successive terms of the divergent series 1/3, 1/4, 1/5...........$1/i$........... are found to be less than those of the series e^{-1}, $(1-e^{(e-1)})$...$(1-s_i)$.........The sum S_{n-1}^* diverges to infinity for $\alpha>1$, and consequently S_{n-1} diverges to minus infinity. The power series $f_1(s_n)$ is a monotonically increasing sequence. Since $f_{n+1}(0)$ is a product of two monotonic functions one increasing and the other decreasing $f_{n+1}(0)$ is not a decreasing function for all values of α and β. It will be monotonically increasing or decreasing according as

$$\frac{f_{n+1}(0)}{f_n(0)}=e^{\beta(S_{n-1}-S_{n-2})}\cdot\frac{f_1(s_n)}{f_1(s_{n-1})}$$

$$=e^{\beta(S_{n-1}-S_{n-2})}\cdot e^{(\alpha+\beta)(s_n-s_{n-1})}$$

is greater than or less than 1 (*i.e.*) according as

$$\beta(s_{n-1})+(\alpha+\beta)(s_n-s_{n-1})>0 \text{ or } <0 \qquad ...(2)$$

(*a*) For $\alpha>0$, $\beta>0$ we have from the inequalities (2)

$$(\alpha+\beta)s_n \gtrless \alpha\, s_{n-1}+\beta \qquad ...(2)'$$

To show that (2)′ holds true for $(n+1)$, consider the recurrence relation,

$$s_{n+1}=e^{\alpha(s_n-1)}$$

Substituting from (2)′

$$s_{n+1}\gtrless e^{\alpha\left\{\frac{\alpha s_{n-1}+\beta}{\alpha+\beta}-1\right\}}$$

$$\gtrless e^{[\alpha(s_{n-1}-1)]\alpha/(\alpha+\beta)}$$

$$\gtrless(s_n)^{\alpha/(\alpha+\beta)} \qquad ...(3)$$

As $s_n < 1$, it can be shown [3] that

$$1-\frac{\alpha}{(\alpha+\beta)} s_n^{-\beta/(\alpha+\beta)}(1-s_h) < s_n^{\alpha/(\alpha+\beta)} < 1-\frac{\alpha}{(\alpha+\beta)}(1-s_n)$$

Inequalities (3) then reduce to

$$s_{n+1} > 1-\frac{\alpha}{(\alpha+\beta)} s_n^{-\beta/(\alpha+\beta)}(1-s_n)$$

i.e.,
$$(\alpha+\beta)s_{n+1} > \alpha s_n+\beta+\alpha(1-s_n^{\beta/(\alpha+\beta)})$$

$$> \alpha s_n+\beta$$

or
$$s_{n+1} < s_n^{\alpha/\alpha+\beta}$$

$$< \frac{(\alpha s_n+\beta)}{\alpha+\beta}$$

Therefore inequalities (2) will be true according as

$$\beta(s_0-1)+(\alpha+\beta)(s_1-s_0) > 0 \text{ or } < 0$$

that is
$$(\alpha+\beta)\, e^{-\alpha}-\beta > 0 \text{ or } < 0$$

(*b*) For $\alpha=0$, $\beta > 0$, $f_n(0)$ will be decreasing.

(*e*) For $\alpha > 0$, $\beta=0$, $f_n(0)$ will be increasing.

Allowing increaments by .05 to α and β, ranges of α and β can be found for which inequalities are true. It is found that for $\alpha=1$, β should not be less than .60 for $f_n(0)$ to be a decreasing function. To be so, β is smaller than .60 for $\alpha > 1$. Since monotonic function $f_{n+1}(0)$ is bounded by 1 above and by 0 below, it will attain a limit. Also for $\alpha < 1$ the equation $x=e^{\alpha(x-1)}$ has only one root $l=1$ [and not two [1]] and for $\alpha > 1$ the equation has two roots $l_1=1$ and $l_2 < 1$

If further $s_n \to l$ and $S_n \to k$ then,

(*i*) $\underset{n\to\infty}{\text{Lt}}\ f_{n+1}(0)=e^{\beta(k-1)(\alpha+\beta)(l-1)}$ for $0 < \alpha < 1$, $\beta > 0$

(ii) $\underset{n\to\infty}{\mathrm{Lt}}\, f_{n+1}(0)=e^{-\beta}$ for $\alpha=0,\ \beta>0$

(iii) $\underset{n\to\infty}{\mathrm{Lt}}\, f_{n+1}(0)=l$ for $\alpha>0,\ \beta=0$

(iv) $\underset{n\to\infty}{\mathrm{Lt}}\, f_{n+1}(0)=0$ for $\alpha\geqslant 1,\ \beta>0$

Table 1 gives for different values of α limiting values of K for $\alpha<1$. For $\alpha\geqslant 1$, k tends to $-\infty$.

TABLE 1

α	K	α	K	α	K	α	K
·05	−·05133	·30	−·36566	·55	−0·86501	·80	−1·93679
·10	−·10569	·35	−·44536	·60	−1·00825	·85	−2·36238
·15	−·16361	·40	−·53322	·65	−1·17587	·90	−2·99745
·20	−·22574	·45	−·63090	·70	−1·37619	·95	−4 16367
·25	−·29280	·50	−·74054	·75	−1·62235	—	—

4. LIMITING GENERATING FUNCTION

The form of the generating function as thc number of generations tends to infinity is determined by the limits of $s_n(s)$ and $S_n(s)$. For all values of α and $0<s<1$, it can be shown [2] that

$$\underset{n\to\infty}{\mathrm{Lt}}\ s_n(s)=l$$

For the convergence of $S_n(s)$ we define the following power series expansion for $s_i(s)$, $0<s\leqslant 1$

$$s_0(s)=s$$

$$s_1(s)=k_{10}+k_{11}s+k_{12}s^2+\dots+k_{1r}s^r+\dots$$

$$s_2(s)=k_{20}+k_{21}s+k_{22}s^2+\dots+k_{2r}s^r+\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$s_n(s)=k_{n0}+k_{n1}s+k_{n2}s^2+\dots+k_{nr}s^r+\dots$$

Subtracting 1 from both sides and summing

$$S_n(s)=\sum_{i=0}^{n}k_{i0}-n-1+\sum_{i=0}^{n}k_{i1}s+\sum_{i=0}^{n}k_{i2}s^2+\dots+\sum_{i=0}^{n}k_{ir}s^r+\dots$$

Further since $s_n(s)=e^{\alpha[s_{n-1}(s)-1]}$, taking first derivative we have

$$s^1{}_n(s)=\alpha\, e^{\alpha[s_{n-1}(s)-1]}\,.\, s^1{}_{n-1}(s)=\alpha s_n(s)s^1{}_{n-1}(s).$$

Differentiating it $(r-1)$ times

$$s_n{}^r(s)=\alpha\{s_n{}^r{}_{-1}(s)+c_1{}^{r-1}s^1{}_n(s).s_{n-1}{}^{r-1}(s)+\ldots+c_{r-1}{}^{r-1}s_n{}^{r-1}(s)\} \quad \ldots(4)$$

Now defining $s_n{}^r(o)=s_n{}^r$ it can be shown that the sequence $s_n{}^r$, $s_n{}^{r-1}$, $s_n{}^2$, $s_n{}^2$ is a decreasing or increasing sequence according as $\alpha\geqq 1$ or $\alpha<1$. For,

when $\alpha\geqq 1$

$$(s_n{}^r-s_n{}^{r-1})=\alpha[s_{n-1}{}^r+c_1{}^{r-1}s^1{}_ns_{n-1}{}^{r-1}+\ldots+c_{r-2}{}^{r-1}s^1{}_{n-1}s_n{}^{r-2}+s_n{}^{r-1}]-s_n{}^{r-1}$$

$$=\alpha[s_{n-1}{}^r+c_1{}^{r-1}s^1{}_ns_{n-1}{}^{r-1}+\ldots+c_{r-2}{}^{r-1}s^1{}_{n-1}s_n{}^{r-2}]+(\alpha-1)s_n{}^{r-1}$$
$$>0.$$

and when $\alpha<1$.

$$s_n{}^r<[s_{n-1}{}^r+\ldots+s_n{}^{r-1}]$$

or

$$s_n{}^r/s_n{}^{r-1}<\{1+c_{r-2}{}^{r-1}s^1{}_{n-1}s_n{}^{r-2}+\ldots+s_{n-1}{}^r/s_n{}^{r-1}]$$
$$<1.$$

Since $s_n{}^r=k_{nr}$

$$k_{i0}<k_{i1}<k_{i2}<\quad<k_{i\,r+1}\ldots \quad \text{for} \quad \alpha\geqq 1.$$

$$k_{i0}>k_{i1}>k_{i2}\quad>k_{i\,r+1}\ldots \quad \text{for} \quad \alpha<1.$$

Similarly to prove the monotonocity of k_{1r}, k_{2r},...k_{nr}, we can proceed by considering the first term in the expression for $s_n{}^r$. That is, for $\alpha\geqq 1$

$$s_n{}^r-s^r{}_{n-1}=\alpha[c_1{}^{r-1}s^1{}_ns_{n-1}{}^{r-3}+c_2{}^{r-2}s_n{}^2s_{n-1}{}^{r-2}+\ldots+s_n{}^{r-1}]+(\alpha-1)s_{n-1}{}^r$$
$$>0$$

and for $\alpha<1$

$$s_n{}^r/s^r{}_{n-1}<1+c_1{}^{r-1}s^1{}_n\,\frac{s_{n-1}{}^{r-1}}{s_{n-1}{}^r}+\ldots\ldots$$
$$<1$$

Therefore k_{1r}, k_{3r},...k_{nr}...decrease or increase monotonically according as $\alpha<1$ or $\alpha\geqq 1$.

For $\alpha<1$, each term of this series is less thau the term of the convergent series k_{10}, k_{20}, ...k_{n0}...

The series $\sum_{n=0}^{\infty} k_{nr}$ is convergent as the auxiliary series is convergent.

Therefore, for all values of $\alpha<1$, $\sum_{r=0} k_{ir}$ is a convergent sum and therefore, $\underset{n\to\infty}{\text{Lt}}\ S_n(s)=S(s)$ for $|s|<1$

The limiting generating function then is

$$\underset{n\to\infty}{\text{Lt}}\ f_n(s)=le^{\beta S(s)}$$

$$=\pi(s)$$

$$=\sum_{i=0}^{\infty} \pi_i s^i \qquad \text{for } 0\leqslant\alpha<1,\ \beta>0$$

$$=l \qquad \text{for } \alpha\geqslant 0,\ \beta=0$$

$$=0 \qquad \text{for } \alpha\geqq 1,\ \beta>0$$

5. Recurrence

A state i is recurrent if and only if starting from state i, within a finite length of time it can again be reached with probability one. For the case when initial size is 1, this will be so if and only if

$$\sum_{n=1}^{\infty} P_{11}{}^{n}=\infty$$

From the relation $f_{n+1}(s)=e^{\beta S_{n-1}(s)}\ f_1(s_n)=e^{\beta S_{n-1}(s)}\ .e^{(\alpha+\beta)(s_n(s)-1)}$ taking 1st derivative and putting $s=0$ we can find $P_{ij}{}^{n}$. Consider the case $j=1$

$$P_{11}{}^{n+1}=e^{\beta S_n+\alpha(s_a-1)}\ [\beta S'_n+\alpha s'_n]$$

$$=e^{\beta\left[\sum^{n} k_{i0}\right]+\alpha(k_{n0}-1)}\left[\beta\sum_{i=0}^{n} k_{i1}+\alpha k_{n1}\right]$$

$$=\beta \sum_{n=1}^{\infty} e^{\beta\left[\sum^{\infty} k_{i0}\right]+\alpha(k_{n0}-1)} \cdot \sum_{\alpha=0}^{n} k_{i1}$$

$$+\alpha \sum k_{n1} e^{\beta\left[\sum^{n} k_{i0}\right]+\alpha(k_{n0}-1)}$$

(*i*) $\alpha>0, \beta>0$

Since k_{io} $(i=1, 2, 3\ldots)$ is an increasing sequence

$k_{i0}>k_{10}$, first term is greater then $\beta \sum_{n=1}^{\alpha} nk_{10}e^{\beta nk_0+\alpha(k_{10}-1)}$

and also second term is greater than

$$\alpha \sum_{n=1}^{\alpha} k_{n1}e^{\beta nk_{10}+\alpha(k_{10}-1)}$$

Both terms are therefore divergent since these are greater than the divergent series. The process is thus recurrent

(*ii*) $\beta=0$.

Only the second term exists which gives

$$\sum P_{11}{}^{n+1}=\alpha \sum_{n=1}^{\infty} k_{n1}e^{\alpha(k_{n1}-1)}$$

and the series is convergent as $k_{n1}<k_{n0}>1$. The sum is finite and the process is transient.

(*iii*) For $\alpha=0$, the first term is again divergent and the process is recurrent. Since all states i, $i>0$ in the process communicate with each other, the process is irreducible. There exist then, $m, n\geqslant 1$ such that

$$P_{ij}{}^{n}>0 \text{ and } P_{ij}{}^{m}>0 \text{ for } i, j>0$$

For $p>0$ $$P_{ji}^{m+n+p} = \sum_{r=0}^{\infty} P_{jr}^{m} P_{rk}^{n+p}$$

$$\geqslant P_{ji}^{m} P_{ik}^{n+p}$$

$$\geqslant P_{ji}^{m} \sum_{r=0}^{\infty} P_{ir}^{n} P_{rj}^{p}$$

$$\geqslant P_{ji}^{m} P_{ij}^{n} P_{ii}^{p}$$

Summing over p, $$\sum_{p=0}^{\infty} P_{jj}^{m+n+p} \geqslant \sum_{p=0}^{\infty} P_{ji}^{m} P_{ii}^{p} P_{ij}^{n}$$

$$\geqslant P_{ji}^{m} P_{ij}^{n} \sum_{p=0}^{\infty} P_{ii}^{p}$$

Hence if $\sum_{p=0}^{\infty} P_{ii}^{p}$ diverges then so will $\sum P_{jj}^{p}$. Therefor all states $i>0$ are recurrent in the process if $\beta>0$. Also, since the period of each state is one, the process is a periodic. We have already shown that lim $P_{ii}^{n}=\pi_i>0$ for all i when $\beta>0$. The process is therefore positive recurrent for $\beta>0$.

Summary

The probability of extinction of the process considered in an earlier paper [1] is further discussed in detail for its monotonocity and the limiting form of the generating function of the process as the number of generations becomes infinitely large is obtained. For $\beta>0$, the process is further shown to be positive recurrent.

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