ON THE CONSTRUCTION OF MUTUALLY ORTHOGONAL LATIN SQUARES OF NON-PRIME-POWER ORDERS

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1. Introduction

If v, the order of a latin square can be expressed as product of u prime powers such that $v = p_1^{n_1} . p_2^{n_2} \ldots p_u^{n_u}$, it is known from MacNeish[4] and Mann[5] that there exists a set of at least $n(v)$ mutually orthogonal latin squares (m.o.l.s.) of order v where

$$n(v) = \min. (p_1^{n_1}, p_2^{n_2}, \ldots, p_u^{n_u})-1.$$

Parker,[6] Bose and Shrikhande[1] and Bose, Shrikhande and Parker[2] improved upon the number $n(v)$ of the above result in some cases while the latter two papers disproved the Euler's conjucture by showing that at least two orthogonal latin squares of order v for $v = 2$ (mod. 4) (except for $v = 6$) exist, using the orthogonal arrays, Rao,[7] and Pairwise Balanced Designs of index unity.

Bose, Chakravarti and Knuth[3] adopted another method of using orthogonal mappings of a group and obtained 5 m.o.l.s. of order 12.

In the present paper, two series of pairwise balanced designs of index unity have been obtained. Through one of these designs the particular case of MaxNeish-Mann result regarding the number of m.o.l.s. of order equal to product of any two primes can be obtained using the methods of Bose and Shrikhande.[1] These designs have further been used to make improvements, in some cases, on the existing lower bound of the maximum possible m.o.l.s. of order v, where v is a neither a prime nor a prime power.

2. Construction of Pairwise Balanced Designs of Index Unity

A pairwise balanced design $D(v;\ k_1, k_2, \cdots k_m)$ of index unity, as defined by Bose and Shrikhande,[1] is an arrangement of v treatments in b blocks such that each block contains either $k_1, k_2, \cdots$ or k_m treatments which are all distinct $(k_i < v;\ k_i \neq k_j)$ and every pair of treat ments occurs in exactly one block of the design. The set of b_i blocks of size k_i is said to be an equiblock component of size k_i.

Consider the affine resolvable BIB design

$$v = s^2, \quad b = s^2 + s, \quad r = s + 1, \quad k = s, \quad \lambda = 1.$$

From one of the $s + 1$ replications of the design delete p blocks and al treatments occurring in those p blocks. Since the blocks of different replications of the above affine resolvable BIB design have one treatment in common, every block of all the replications except the one from which the blocks are deleted looses p treatments which occur in the p deleted blocks. The remaining design is a pairwise balanced design with parameters

$$v = s(s-p), \; b = s^2 + s - p, \; k_1 = s - p, \; k_2 = s, \; \lambda = 1 \quad \text{(I)}$$

In this design series there are s replications with s blocks each of size $s - p$ and a single replication with $s-p$ blocks of size s each.

To the pairwise balanced design I, add a block of $s + 1$ new treatments and to the blocks of each of the $s + 1$ replications add one each of the $s + 1$ new treatments. We then have a pairwise balanced design with parameters

$$v = s(s-p) + s + 1, \quad b = s^2 + s - p + 1,$$
$$k_1 = s - p + 1, \quad k_2 = s + 1, \quad \lambda = 1 \quad \text{(II)}$$

3. CONSTRUCTION OF M.O.L.S.

A pairwise balanced design $D(v; k_1, k_2, \ldots, k_m)$ of index unity is said to be separable if the blocks of every equiblock component $(D_i)\, i = 1, 2, \ldots, m$ can be divided into subsets such that every treatment occurs in a subset exactly k_i times or exactly once. Then we have the

THEOREM 3.1. Let there exist a pairwise balanced design $D(v; k_1, k_2, \ldots, k_m)$ of index unity and suppose there exist $q_i - 1$ m.o.l.s. of order k_i. If

$$q = \min.\,(q_1, q_2, \ldots, q_m)$$

then there exist at least $(q - 2)$ m.o.l.s. of order v. If the design (D) is separable then the number of m.o.l.s. of order v is at least $(q - 1)$.

The proof, given by Bose and Shrikhande,[1] follows by showing that we can construct an orthogonal array $[v^2, q, v, 2]$ from the design (D)

and if it is separable we can have the orthogonal array $[v^2, q+1, v, 2]$. Co-ordinatising any two rows of the array we have $(q-1)$ m.o.l.s. of order v.

Further in the design $D(v; k_1, k_2, \ldots k_m)$ the set of equiblock components $(D_1), (D_2), \ldots (D_l)$, $l < m$, will be said to be a clear set if the $\sum_{i=1}^{l} b_i$ blocks comprising $(D_1), (D_2), \ldots, (D_l)$ are disjoint, that is, no two blocks contain a common treatment. Then, Bose, Shrikhande and Parker[2] proved the

THEOREM 3.2. Let there exist a pairwise balanced design $D(v; k_1, k_2, \ldots, k_m)$ of index unity such that a set of equiblock components $(D_1), (D_2), \ldots, (D_l)$, $l < m$ is a clear set. If there exist $(q-1)$ m.o.l.s. of order k_i and if

$q^* = \min. (q_1 + 1, q_2 + 1, \ldots q_l + 1, q_{l+1}, \ldots, q_m)$ then there exist at least $q^* - 2$ m.o.l.s of order v.

In this case the proof follows by showing that we can construct an orthogonal array $[v^2, q^*, v, 2]$.

Now, the series I of Section 2 is separable since the equiblock component (D_1) of size $k_1 = s - p$ is divided into s subsets (replications) such that each treatment occurs once and only once in each subset and the equiblock component (D_2) of size $k_2 = s$ is a single such subset. Hence from Theorem 3.1, we have the particular case of MacNeish-Mann result for the product of any two primes that

$$N[s(s-p)] \geqslant N(s-p)$$

where $N(v)$ denotes the maximum possible number of m.o.l.s. of order v. The results of MaxNeish-Mann for the product of any two primes have been obtained earlier through other methods but all of them could not be obtained through the methods of Bose, Shrikhande and Parker.

We shall give here some examples to illustrate the use of these design series in the construction of m.o.l.s. of order v in whose case an improvement in the lower bound of $N(v)$ is possible.

Example 1.—Consider the design $D(100; 9, 12)$ obtained from Series II for $s = 11$, $p = 3$. Take out from the design the 100th treatment (the treatment which is replicated only 9 times) and then 6 treatments and 3 treatments from two different blocks of the equiblock component of size 12. We have a design $D(90; 5, 7, 8, 9, 11)$ where the equiblock component consisting of the single block of size 5 is

v	Existing l.b. for $N(v)$	Improved l.b. for $N(v)$	s	p	x	Series No.	Remarks
90	2	4	11	3	−10	II	Example 1
94	2	5	11	3	− 6	II	Delete the 100th, 3 and 2 treatments respectively from two blocks of size 12 as in Example 1
106	2	5	13	4	−11	I	6 and 5 treatments respectively from two blocks of size 13 as in Example 1
110	2	5	13	4	− 7	I	5 and 2 treatments respectively from two blocks of size 13 as in Example 1
114	2	5	13	4	− 3	I	2 and 1 treatments respectively from two blocks of size 13 as in Example 1
116	3	5	13	4	− 1	I	Delete any treatment
124	4	5	16	9	+12	I	Example 2
134	5	6	16	8	−11	II	Example 3
138	5	6	16	8	− 7	II	As in Example 3
140	5	6	16	8	− 7	II	As in Example 3
142	5	6	17	9	−12	II	11 treatments including the treatment occurring 9 times from the last block and one from other treatments
146	5	6	17	8	− 7	I	6 and 1 treatments from two blocks of size 17
148	3	4	13	2	+ 5	I	As in Example 2
150	5	6	17	9	− 4	I	Example 4
154	2	4	13	1	− 2	I	Any two treatments from a block of size 13

clear. Hence from Theorem 3.2 above, we have $N(90) \geqslant 4$. The existing lower bound in this case is only 2.

Example 2.—To the design $D(112;\ 7, 16)$ for $s = 10$, $p = 9$ in Series I, add 12 treatments, one to each of 12 sets and a block of size 12. Since the single block of size 12 is clear, $N(124) \geqslant 5$, the known lower bound in this case being 4.

Example 3.—Consider the design $D(145;\ 8, 17)$ of Series II for $s = 16$, $p = 8$. Delete from it any 10 treatments from the new block except that which is replicated only 9 times and one more treatment from the rest (not occurring in the new block). We have the design $D(134;\ 8, 9, 16, 17)$ in which the equiblock component of size 7 is a clear set. Hence $N(134) \geqslant 6$, the known lower bound being 5.

Example 4.—From the design $D(154;\ 9, 18)$ for $s = 17$, $p = 9$ in Series II, delete two treatments from the new block one of which is replicated 9 times; and two more treatments not coming from the last block and which do not occur together in any block of the two sets with which the previous two new treatments occur. We then have $D(150;\ 7, 8, 9, 16, 17)$. Since the equiblock component of size 7 is clear, $N(150) \geqslant 6$, the known lower bound being 5.

In the following table the values of v ($v \leqslant 154$) for which the lower bound of $N(v)$ could be improved over the existing lower bound are given. The positive and negative signs for the values of x denote respectively the addition and deletion of that number of treatments from the design. Under the remarks, the method of addition or deletion is indicated.

4. Summary

Two new series of pairwise balanced designs of index unity have been obtained. Through one of them the particular case of the MacNeish-Mann result on the maximum possible m.o.l.s. of order equal to product of any two primes can be obtained. Some improvements on the lower bound of maximum possible m.o.l.s. of non-prime-power orders have been made.

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