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Optimal Asymmetric Fractional Factorial Plans using Finite Projective Geometry

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SUMMARY

Dey *et al.* (2005) obtained universally optimal plans for asymmetric factorial experiments under hierarchical model that includes the mean, all M.E. and a specified set of 2FI, assuming other interactions are negligible. In this paper, we construct two new asymmetric optimal fractional factorial plans under new hierarchical models using finite projective geometry. One of the optimal plans permits the estimability of the mean, all M.E., a specified set of 2FI and a specified set of 3FI. The other plan permits the estimability of the mean, all M.E. and a specified set of 2FI.

We also construct some new asymmetric optimal fractional factorial plans under a hierarchical model for estimation of the mean, all M.E. and a specified set of 2FI.

Key words: Galois field, Finite projective geometry, Universal optimality, Saturated plans.

1 INTRODUCTION

Fractional factorial designs are commonly used in industrial experiments where a large number of factors has to be studied. Optimality of fractional factorial plans has been studied by many researchers in recent years. In practical situation, all factorial effects involving same number of factors may not be equally important. The issue of estimability and optimality in the context of two level factorials has been studied by Hedayat and Pesotan (1992, 1997), Wu and Chen (1992), Chiu and John (1998) and Ke and Tang (2003). Optimality results for arbitrary factorials including asymmetric ones were obtained by Dey and Mukerjee (1999). Dey *et al.* (2005) (we will abbreviate in the remainder as DSD) obtained optimal asymmetrical fractional factorial plans for estimation of the mean, all M.E. and a specified set of 2FI using finite projective geometry. In this paper, we construct two new asymmetric optimal fractional factorial plans under new hierarchical models using finite projective geometry. One of the optimal plans permits the estimability of the mean, all M.E., a specified set of 2FI and a specified set of 3FI. The other plan permits the estimability of the mean, all M.E. and a specified set of 2FI.

We also construct some new asymmetric optimal fractional factorial plans under a hierarchical model for estimation of the mean, all M.E. and a specified set of 2FI. In Section 2, we give some preliminaries of finite projective geometry.

In Section 3.1, we construct a new optimal fractional factorial plan for an $(m^{u_1}) \times m^{A+B}$ experiment which permits the estimation of the mean, all M.E. a specified set of 2FI and a specified set of 3FI. In Section 3.2, we construct new optimal fractional factorial plans for an $(m^3) \times (m^2) \times m^w$, $w > 1$ experiment which permits the estimation of the mean, all M.E. and a specified set of 2FI. We also construct some new optimal fractional factorial plans for an $(m^d)^e \times (m^g)^h$ experiment which permits the estimation of the mean, all M.E. and a specified set of 2FI in Section 3.3.

2. FINITE PROJECTIVE GEOMETRY

A finite projective geometry of (r – 1) dimension PG(r – 1, m) over GF(m), Galois field of order m, m is a prime power, consists of the ordered set $(x_0, x_1, \ldots, x_{r-1})$ of points where x_i (i = 0, 1, ... , r – 1) are elements of GF(m) and all of them are not simultaneously zero. For any $\lambda \in GF(m)$ $(\lambda \neq 0)$, the point $(\lambda_0, \ldots, \lambda_{r-1})$

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represents the same point as that of $(x_0, ..., x_{r-1})$. The total number of points lying on PG(r – 1, m) is $\frac{m^r - 1}{m - 1}$. All those points which satisfy a set of (r – t – 1) linearly independent homogeneous equations with coeffcients from GF(m) (all of them are not simultaneously zero within the same equation) is said to represent a t-flat in PG(r – 1, m).

In particular a 0-flat, a 1-flat, ..., and a (r – 2)-flat respectively in PG(r – 1, m) are known as a point, a line ... and a hyperplane of PG(r – 1, m). The number of points lying on a (t – 1)-flat is $\frac{m^t - 1}{m - 1}$ but the number of independent points lying on a (t – 1)-flat is t. The number of (r – 2)-flats within a PG(r – 1, m) which contain a given (r – 3)-flat is (m + 1). One may refer to Hirschfeld (1998) for more details.

3. CONSTRUCTION OF OPTIMAL PLANS

Consider a factorial experiment involving n factors $F_1, F_2, ..., F_n$, where the factor F_i (i = 1, 2, ..., n) has m^{t_i} (i = 1, 2, ..., n) levels, m is a prime power and t_i (i = 1, 2, ..., n) is a positive integer. We use (r – 1) dimensional finite projective geometry PG(r – 1, m) over GF(m) to construct m^r-run plans, r being an integer. DSD have introduced the following three models.

$M_1 : (\mu, F_1, ..., F_{2u}, F_1F_2, F_3F_4, ..., F_{2u-1}F_{2u})$

$M_2 : (\mu, F_1, ..., F_{u+v}, F_iF_j, 1 \le i \le u, u + 1 \le j \le v)$

$M_3 : (\mu, F_1, ..., F_u, F_1F_2, F_2F_3, ..., F_{u-1}F_u, F_uF_1)$

A plan d which is universally optimal under the above model will be denoted by $d \equiv (F_1, F_2; F_3, F_4; ... ; F_{2u-1}, F_{2u})_1$, $d \equiv (F_1, ..., F_u; F_{u+1}, ..., F_{u+v})_2$ and $d \equiv (F_1, ..., F_u)_3$ respectively.

Here we will introduce the following notations to specify two new models

M4 : $(\mu, F_1, ..., F_{2u+1}, F_iF_j$, for any $2 \le i \le u + 1$, for all $u + 2 \le j \le 2u + 1)$

M5 : $(\mu, F_1, ..., F_{u+v+1}, F_1F_i, 2 \le i \le u + v + 1, F_jF_k, F_1F_jF_k, 2 \le j \le u + 1, u + 2 \le k \le u + v + 1)$

A plan d which is universally optimal under the above models will be denoted by $d \equiv (F_1 : F_2, ..., F_{u+1} : F_{u+2}, ..., F_{2u+1})_4$ and $d \equiv (F_1; F_2, ..., F_{u+1}; F_{u+2}, ..., F_{u+v+1})_5$ respectively.

All effects which are not included in the model are assumed to be negligible. Here μ denotes the mean, F_i denotes the M.E. of the ith factor, $F_i F_j$ denotes the interaction between F_i and F_j, and $F_iF_jF_k$ denotes the interaction between F_i, F_j and F_k.

Throughout this paper, we assign factors F_0, F_1 ... etc. to a (t – 1)-flat in PG(r – 1, m), where t points are independent and we assign m^t levels to each of these factors. We also assign m levels to each of the factors G_0, G_1, ... etc.

3.1 Construction of Optimal Plan for an $(m^{u_1}) \le m^{A+B}$, $u_1 > 1$ Experiment

Here we extend Theorem 2.2 given in DSD to construct an optimal asymmetric fractional factorial plan for estimation of the mean, all M.E., a specified set of 2FI and a specified set of 3FI using finite projective geometry.

Theorem 3.1. Let $F_1, ..., F_f$ be f factors of a factorial experiment. For any prime power m, the factor F_u has m^{t_u} levels where t_u is a positive integer and u = 1, 2, ..., f. We assign the f M.E. $F_1, ..., F_f$, the k, 2FI $F_{i1}F_{j1}, ..., F_{ik}F_{jk}$ and p, 3F$_{i1}$F$_{j1}$F$_{q1}$, ..., $F_{ip}F_{jp}F_{qp}$ to the points in PG(r – 1, m) as given below.

(1) the factor F_i to a $(t_i - 1)$-flat for i = 1, 2, ..., f in PG(r – 1, m), these flats being disjoint for F_i, F_j, $i \ne j$.

(2) the 2FI F_iF_j is assigned to the

$$\left(\frac{(m^{t_i} - 1)}{(m-1)}\frac{(m^{t_i} - 1)}{(m-1)}\right)(m-1) = \frac{(m^{t_i} - 1)(m^{t_j} - 1)}{m - 1}$$

points in the $(t_i + t_j - 1)$-flat through the $(t_i - 1)$-flat F_i and the $(t_j - 1)$-flat F_j but not in F_i and F_j.

(3) the 3FI, $F_iF_jF_q$ is assigned to the

$$\left(\frac{(m^{t_i} - 1)}{(m-1)}\frac{(m^{t_j} - 1)}{(m-1)}\frac{(m^{t_q} - 1)}{(m-1)}\right) \times (m-1)^2$$

$$= \frac{(m^{t_i} - 1)(m^{tj} - 1(m^{t_q} - 1)}{m - 1}$$ points in the $(t_i + t_j + t_q - 1)$-flat through the $(t_i - 1)$-flat F_i, the

$(t_j - 1)$-flat F_j and the $(t_q - 1)$-flat F_q but not in F_i, F_j, F_q, F_iF_j, F_iF_q and F_jF_q.

If $$\sum_{u=1}^{f}\left(\frac{m^{t_u}-1}{m-1}\right)+\sum_{u=1}^{k}\frac{(m^{t_{i_u}}-1)(m^{t_{j_u}}-1)}{(m-1)}$$
$$+\sum_{u=1}^{p}\frac{(m^{t_{i_u}}-1)(m^{t_{j_u}}-1)(m^{t_{q_u}}-1)}{(m-1)}$$

points corresponding to $F_1, ..., F_f, F_{i_1}, ..., F_{j_k}$, $F_{j_k}, F_{i_1} F_{q_1}, ..., F_{i_p}F_{j_p}F_{q_p}$ are all distinct points of PG(r – 1,m), then we can obtain a universally optimal saturated plan (we will abbreviate it in the remainder as UOSP) for estimation of the M.E. $F_1, ..., F_f$, k 2FI $F_{i_1}F_{j_1}, ..., F_{i_k}F_{j_k}$ and p 3FI $F_{i_1}F_{j_1}F_{q_1}, ..., F_{i_p}F_{j_p}F_{q_p}$ involving m^r runs.

Proof. Let F_u be an $r \times t_u$ matrix with the t_u column vectors corresponding to tu independent points in $(t_u - 1)$-flat F_u.

Then the plan can be generated by the row space of the $r \times \sum_{u=1}^{f} t_u$ matrix $P = [F_1 : ..... : F_f]$, where t_u columns of F_u represent the levels of the factor F_u and each element of the row space P represents a run in the plan. To prove that the plan is UOSP, it suffces to show that the following matrices have full column rank.

(a) $[F_u : F_v]$, $1 \le u < v \le f$

(b) $[F_u : F_{i_v} : F_{j_v}]$, $1 \le u \le f$, $1 \le v \le k$

(c) $[F_{i_u} : F_{j_u} : F_{i_v}, F_{j_v}]$, $1 \le u < v \le k$

(d) $[F_u : F_{i_c} : F_{j_c} : F_{q_c}]$, $1 \le u \le f$, $1 \le c \le p$

(e) $[F_{i_v} : F_{j_v} : F_{i_c} : F_{j_c} : F_{q_c}]$, $1 \le v \le k$, $1 \le c \le p$

(f) $[F_{i_b} : F_{j_b} : F_{q_b} : F_{i_c} : F_{j_c} : F_{q_c}]$, $1 \le b < c \le p$

(g) $[F_u : F_{i_v} : F_{j_v} : F_{i_c} : F_{j_c} : F_{q_c}]$, $1 \le u \le f$, $1 \le v \le k$, $1 \le c \le p$

where a matrix F_u $(1 \le u \le f)$ appears only once if it is repeated in (b),... or (g).

DSD have proved that matrices in (a) to (c) are all full column rank.

In (d), there are two cases.

Case 1. If $u = i_c$ or j_c or q_c, then the matrix (d) reduces to $[F_{i_c} : F_{j_c} : F_{q_c}]$, which has full column rank.

Case 2. If u, i_c, j_c and q_c are all distinct, then the $(t_u - 1)$-flat F_u and the $(t_{i_c} + t_{j_c} + t_{q_c} - 1)$-flat consisting of points $F_{i_c}, F_{j_c}, F_{q_c}, F_{i_c}F_{j_c}, F_{i_c}F_{q_c}, F_{j_c}F_{q_c}$ and $F_{i_c}F_{j_c}F_{q_c}$ are disjoint. Hence the columns of F_u are independent of columns of $[F_{i_c} : F_{j_c} : F_{q_c}]$. Thus the matrix $[F_u : F_{i_c} : F_{j_c} : F_{q_c}]$ has full column rank.

In (e), there are three cases.

Case 1. If $i_v = i_c$ or j_c or q_c and $j_v = i_c$ or j_c or q_c where i_v and j_v are distinct, then the matrix in (e) reduces to $[F_{i_c} : F_{j_c} : F_{q_c}]$, which has full column rank.

Case 2. If $i_v = i_c$ or j_c or q_c, or, $j_v = i_c$ or j_c or q_c, then the matrix in (e) reduces to $[F_{j_v} : F_{i_c} : F_{j_c} : F_{q_c}]$, which has full column rank.

Case 3. If i_v, j_v, i_c, j_c and q_c are all distinct then the $(t_{i_v} + t_{j_v} - 1)$-flat consisting of points F_{i_v}, F_{j_v} and $F_{i_v}F_{j_v}$ and the $(t_{i_c} + t_{j_c} + t_{q_c} - 1)$-flat consisting of points $F_{i_c}, F_{j_c}, F_{q_c}, F_{i_c}F_{j_c}, F_{i_c}F_{q_c}, F_{j_c}F_{q_c}$ and $F_{i_c}F_{j_c}F_{q_c}$ are disjoint. Hence the columns of $[F_{i_v} : F_{j_v}]$ are independent of columns of $[F_{i_c} : F_{j_c} : F_{q_c}]$. Thus the matrix $[F_{i_v} : F_{j_v} : F_{i_c} : F_{j_c} : F_{q_c}]$ has full column rank.

In (f), there are three cases.

Case 1. If any two out of ib, jb and qb are same as any two out of i_c, j_c and q_c, the matrix in (f) reduces to $[F_{q_b} : F_{i_c} : F_{j_c} : F_{q_c}]$, which has full column rank.

Case 2. Any one out of i_b, j_b and q_b is same as any one out of i_c, j_c and q_c, the matrix in (f) reduces to $[F_{j_b} : F_{q_b} : F_{i_c} : F_{j_c} : F_{q_c}]$ and hence the matrix has full column rank.

Case 3. If i_b, j_b, q_b, i_c, j_c and q_c are all distinct then the $(t_{i_b} + t_{j_b} + t_{q_b} - 1)$-flat consisting of $F_{i_b}, F_{j_b}, F_{q_b}, F_{i_b}F_{j_b}, F_{i_b}F_{q_b}, F_{j_b}F_{q_b}$ and $F_{i_b}F_{j_b}F_{q_b}$ and the $(t_{i_c} + t_{j_c} + t_{q_c} - 1)$-flat consisting of $F_{i_c}, F_{j_c}, F_{q_c}, F_{i_c}F_{j_c}, F_{i_c}F_{q_c}, F_{j_c}F_{q_c}$ and $F_{i_c}F_{j_c}F_{q_c}$ are disjoint. Hence the columns of $[F_{i_b} : F_{j_b} : F_{q_b}]$ are independent of columns of $[F_{i_c} : F_{j_c} : F_{q_c}]$. Thus the matrix in (f) has full column rank.

In (g), there are four cases.

Case 1. If $u = i_v$ or j_v and also i_v and j_v are same as any two out i_c, j_c and q_c, the matrix in (g) reduces to $[F_{i_c} : F_{j_c} : F_{q_c}]$. Hence the matrix has full column rank.

Case 2. If $u = i_v$ or j_v and any one out of i_v and j_v is same as any one out of i_c, j_c and q_c, then matrix in (g) reduces to $[F_{j_v} : F_{i_c} : F_{j_c} : F_{q_c}]$. Hence the matrix has full column rank.

Case 3. If $u = i_v$ or j_v or i_c or j_c or q_c or, any one out of i_v and j_v is same as any one out of i_c, j_c and q_c, then the matrix in (g) reduces to $[F_{i_v} : F_{j_v} : F_{i_c} : F_{j_c} : F_{q_c}]$ or $[F_u : F_{j_v} : F_{i_c} : F_{j_c} : F_{q_c}]$. Thus the matrix has full column rank.

Case 4. If u, i_v, j_v, i_c, j_c and q_c are all distinct, then the $(t_u - 1)$-flat F_u, the $(t_{i_v} + t_{j_v} - 1)$-flat consisting of $F_{i_v}, F_{j_v}, F_{i_v}F_{j_v}$ and the $(t_{i_c} + t_{j_c} + t_{q_c} - 1)$-flat consisting of $F_{i_c}, F_{j_c}, F_{q_c}, F_{i_c}F_{j_c}, F_{i_c}F_{q_c}, F_{j_c}F_{q_c}$ and $F_{i_c}F_{j_c}F_{q_c}$ are disjoint. Hence the columns of F_u, columns of $[F_{i_v} : F_{j_v}]$ and columns of $[F_{i_c} : F_{j_c} : F_{q_c}]$ are mutually independent. Thus the matrix has full column rank.

Lemma 3.1. If $r = u + v + w$, then $\exists$ a $(u - 1)$-flat, a$(v - 1)$-flat and a $(w - 1)$-flat which are disjoint in PG$(r - 1, m)$, where r, u (≥ 1), $v(\geq 1)$, and $w(\geq 1)$ are integers.

Using Lemma 3.1 and Theorem 3.1 we derive the following theorem.

Theorem 3.2. For any prime power m, one can construct a UOSP d for an $(m^{u_1}) \times m^{A+B}$ experiment involving m^r $(r \geq 5)$ runs for estimation of the mean, $(A + B + 1)$ M.E. F_0, G_i $(i = 1, ..., A + B)$, $(A + B + AB)$ 2FI F_0G_i $(i = 1, ..., A + B)$, G_jG_k $(j = 1, ..., A; k = A + 1, ..., A + B)$ and AB 3FI $F_0G_jG_k$ $(j = 1, ..., A; k = A + 1, ..., A + B)$ where

$$d \equiv (F_0; G_1, ..., G_A; G_{A+1}, ..., G_{A+B})_5$$

Proof. Let K_i be a $(u_i - 1)$-flat $(i = 1, 2, 3)$ in PG$(r - 1, m)$ such that $u_1 + u_2 + u_3 = r$.

The number of points in K_i is $\frac{m^{u_i} - 1}{m - 1}$ for $i = 1, 2, 3$.

Without loss of generality, let us assign the points of

(i) K_1 as F_0 where factor F_0 is at m^{u_1} level, u_1 is the number of independent points in $(u_1 - 1)$-flat K_1.

(ii) K_2 as $G_1, G_2, ..., G_A$, since the number points in K_2 is $\frac{m^{u_2} - 1}{m - 1}$ (= A say). Factors $G_1, ..., G_A$ are at m level each.

(iii) K_3 as $G_{A+1}, G_{A+2}, ..., G_{A+B}$, since the number of points in K_3 is $\frac{m^{u_3} - 1}{m - 1}$ (= B say). Factors $G_{A+1}, ..., G_{A+B}$ are at m level each.

Let $P = [F_0 \mid G_1, ..., G_A \mid G_{A+1}, ..., G_{A+B}]$. It can be verified that the matrix P satisfies the condition of Theorem 3.1.

Now we can obtain a UOSP d generated by the row space of the matrix P for an $(m^{u_1}) \times m^{A+B}$ experiment involving mr runs for estimation of the mean, $(A + B + 1)$ M.E. F_0, G_i $(i = 1, ..., A + B)$, $(A + B + AB)$ 2FI F_0G_i $(i = 1, ..., A + B)$, G_jG_k $(j = 1, ..., A; k = A + 1, ..., A+B)$ and AB 3FI $F_0G_jG_k$ $(j = 1, ..., A; k = A + 1, ...; A + B)$.

Example 3.1. Consider PG(5, 2) over GF(2)

Here $r = 6$, $m = 2$. Let $u_1 = u_2 = u_3 = 2$

Thus K_i $(i = 1, 2, 3)$ is a 1-flat (line) in PG(5, 2)

The number of points in K_i $(i = 1, 2, 3)$ is $\frac{2^2 - 1}{2 - 1} = 3$

These three flats, i.e. K_1,K_2 and K_3 are disjoint. The points of K_1,K_2 and K_3 are

K_1 : (001000, 000110, 001110)

K_2 : (010110, 101110, 111000)

K_3 : (011101, 101111, 110010)

Let us assign the points of

(i) K_1 as F_0, where factor F_0 is at 2^2 level.

(ii) K_2 as G_1,G_2,G_3. The factors G_1,G_2 and G_3 are at 2 level each.

(iii) K_3 as G_4,G_5,G_6. The factors G_4,G_5 and G_6 are at 2 level each.

Let

$$P = [F_0 \mid G_1G_2G_3 \mid G_4G_5G_6] = \left[\begin{array}{c|ccc|ccc} 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Now we can obtain an optimal plan d = (F_0; $G_1G_2G_3$;$G_4G_5G_6)_5$ generated by the row space of the matrix P for a 4×2^6 experiment involving 2^6 runs for estimation of the mean, 7 M.E. F_0,G_i (i = 1, ... , 6) 15 2FI F_0G_i (i = 1, ... , 6), G_jG_k (j = 1, 2, 3, k = 4, 5, 6) and 9 3FI $F_0G_jG_k$ (j = 1, 2, 3, k = 4, 5, 6).

Table 3.1

S. No.	PG(r–1, m)	(u_1, u_2, u_3)	Optimal plan	Experi-ment
1	PG(6,2)	(2, 2, 3)	$d \equiv (F_0; G_1, ..., G_3; G_4, ..., G_{10})_5$	4×2^{10}
2	PG(6,2)	(3, 2, 2)	$d \equiv (F_0; G_1, ..., G_3; G_4, ..., G_6)_5$	8×2^6
3	PG(6,2)	(3, 3, 1)	$d \equiv (F_0; G_1, ..., G_7; G_8)_5$	8×2^8
4	PG(5,3)	(2, 2, 2)	$d \equiv (F_0; G_1, ..., G_4; G_5, ..., G_8)_5$	9×3^8

3.2 Construction of Optimal Plans for an $(m^3) \times (m^2) \times m^w$, w > 1 Experiment

We now construct the optimal asymmetric fractional factorial plans for an $(m^3) \times (m^2) \times m^w$, w > 1 experiment for estimation of the mean, all M.E. and specified set of 2FI using PG(r – 1, m) over GF(m), m is a prime power.

Lemma 3.2. If r = c + (r – c), then ∃ a (c – 1)-flat and an (r – c – 1)-flat which are disjoint in PG(r – 1,m).

Using Lemma 3.2 we derive the following theorems.

Theorem 3.3. For any prime power m, one can construct a UOSP d for an $(m^3) \times (m^2) \times (m^w)$ (w > 1) experiment involving mr (r ≥ 6) runs for estimation of the mean, (w + 2) M.E. F_0, F_1,G_1, ... ,G_w and (w + 1) 2FI F_0F_1, F_0G_1, ... , F_0G_w where d ≡ (F_0; F_1,G_1, ... ,$G_w)_2$.

Proof. Using Lemma 3.2 we derive the following cases. In case I, we consider c = 2 as well as in case II, c = 3.

Case I. Let F_0 be a line in PG(r – 1,m). K is an (r – 3)-flat in PG(r – 1,m). F_0 and K are disjoint. Let F_1 be a plane contained in K. G_1, G_2 ...G_w (where $w = m^3\left(\frac{m^{r-5}-1}{m-1}\right)$ are the points in K disjoint from F_1.

Here, factor F_0 is at m^2 level, factor F_1 is at m^3 level and G_1, ... ,G_w are each at m level.

$$P = [F_0 \mid F_1, G_1, ..., G_w]$$

Now, one can obtain a UOSP d generated by the row space of the matrix P for an $(m^2) \times (m^3) \times m^w$ experiment involving mr(r l 6) runs for estimation of the mean, (w + 2) M.E. F_0, F_1,G_1, ... ,G_w and (w + 1) 2FI F_0F_1, F_0G_1, ... F_0G_w where d ≡ (F_0; F_1,G_1, ...$G_w)_2$.

Example 3.2. Consider PG(5, 2) over GF(2). F_0 is a line. Let the points of F_0 be (001000, 101000, 100000). K is a 3-flat disjoint from F_0. Let the points of K be (000001, 000010, 000011, 000100, 000101, 000110, 000111, 010000, 010001, 010010, 010011, 010100, 010101, 010110, 010111). F_1 is a plane contained in K.

The points of F_1 are (000001, 000010, 000011, 000100, 000101, 000110, 000111). G_1,G_2, ... ,G_8 are the points in K disjoint from F_1. The factors F_0 and F_1 are at 2^2 level and at 2^3 level respectively. G_1, ...,G_8 are at 2-level each.

$P = [F_0 \mid F_1\ G_1 \ldots G_8]$

$$= \left[\begin{array}{c|ccccccccc} F_0 & F_1 & G_1 & G_2 & G_3 & G_4 & G_5 & G_6 & G_7 & G_8 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Now, we can obtain an optimal plan $d \equiv (F_0; F_1, G_1 \ldots G_8)_2$ generated by the row space of the matrix P for a $4 \times 8 \times 28$ experiment involving 2^6 runs for estimation of the mean, 10 M.E. $F_0, F_1, G_1, \ldots G_8$ and 9 2FI $F_0F_1, F_0G_1, \ldots F_0G_8$.

Table 3.2

Sl.No.	PG(r – 1,m)	Optimal plan	Experiment
1.	PG(5,2)	$d \equiv (F_0; F_1, G_1 \ldots G_8)_2$	$4 \times 8 \times 2^8$
2.	PG(6,2)	$d \equiv (F_0; F_1, G_1 \ldots G_{24})_2$	$4 \times 8 \times 2^{24}$
3.	PG(5,3)	$d \equiv (F_0; F_1, G_1 \ldots G_{27})_2$	$9 \times 27 \times 3^{27}$

Proof.

Case II. Consider $PG(r - 1, m) (r \geq 6)$ over GF(m). Let F_0 and K be the plane and (r – 4)-flat in PG(r – 1,m) respectively. F_0 and K are disjoint. Let F_1 be a line contained in K. $G_1, G_2 \ldots, G_w$ (where $w = m^2\left(\frac{m^{r-5}-1}{m-1}\right)$ are the points in K disjoint from the points in F_1.

Here, the factors F_0 and F_1 are at m^3 level and at m^2 level respectively since, F_0 is a plane and F_1 is a line. $G_1, \ldots, G_w$ are at m level each.

$$\text{Let } P = [F_0 \mid F_1, G_1, \ldots, G_w]$$

It can be verified that the matrix P satisfies the condition of Theorem 2.2 of DSD.

Now one can obtain a UOSP d generated by the row space of the matrix P for an $(m^3) \times (m^2) \times m^w$ experiment involving $m^r (r \geq 6)$ runs for estimation of the mean, (w + 2) M.E. $F_0, F_1, G_1, \ldots, G_w$ and (w + 1) 2FI $F_0F_1, F_0G_1, \ldots F_0G_w$ where $d \equiv (F_0; F_1, G_1, \ldots, G_w)_2$.

Table 3.3

Sl.No.	PG(r – 1,m)	Optimal plan	Experiment
1.	PG(5, 2)	$d \equiv (F_0; F_1, G_1, \ldots G_4)_2$	$8 \times 4 \times 2^4$
2.	PG(6, 2)	$d \equiv (F_0; F_1, G_1, \ldots G_{12})_2$	$8 \times 4 \times 2^{12}$
3.	PG(5, 3)	$d \equiv (F_0; F_1, G_1, \ldots G_9)_2$	$27 \times 9 \times 3^9$

3.3 Construction of Optimal Plans for an $(m^d)^e \times (m^g)^h$ Experiment

We construct the optimal asymmetric fractional factorial plans for estimation of the mean, all M.E. and specified set of 2FI using finite projective geometry. Here we will make use of Theorem 2.2 of DSD to construct the optimal plans.

Theorem 3.4. For any prime power m, one can construct a UOSP d for an $(m^p) \times m^{2q}$, $p \geq 2$, $q \geq 4$ experiment involving m^r runs $(r \geq 4)$ for estimation of the mean, (2q + 1) M.E. $F_1, G_1, \ldots, G_{2q}$ and q 2FI G_jG_{q+i} for any one j = 1, 2, ..., q and for all i = 1, 2, ..., q where $d \equiv (F_1 : G_1, G_2, \ldots, G_q : G_{q+1}, G_{q+2}, \ldots, G_{2q})_4$.

Proof. Let H_1 and H_2 be two hyperplanes i.e. ((r – 2)-flats) in PG(r – 1,m) which are not disjoint. The number of points in H_1 and H_2 is $\frac{m^{r-1}-1}{m-1}$. They will intersect in an (r – 3)-flat H_3. That is, the common points between H_1 and H_2 are the points of (r – 3)-flat H_3. The number of points of (r – 3)-flats H_3 is $\frac{m^{r-2}-1}{m-1}$

$$|H_1 \cup H_2| = \frac{(2m^{r-1} - m^{r-2} - 1)}{(m-1)} \quad (= f \text{ say}).$$

Let F be a set of f points obtained by taking union of the two hyperplanes H_1 and H_2. Now we divide these f points into three sets such that in Set 1, there will be $\frac{m^{r-2}-1}{(m-1)}$ points, which are common points between the two hyperplanes H_1 and H_2. These are points of (r – 3)-flat H_3 where (r – 2) (= p say) points are independent.

In Set 2 and Set 3, there will be m^{r-2} (= q say) points each obtained by deleting the common points from each of the hyperplanes H_1 and H_2 respectively.

Thus the three sets are disjoint. Now assign the points in the sets as follows.

In Set 1 : F_1 (the factor F_1 is at m^p level, $p = r - 2$)

In Set 2 : $G_1, G_2, \ldots, G_q$ (each factor is at m level, $q = m^{r-2}$)

In Set 3 : $G_{q+1}, G_{q+2}, \ldots, G_{2q}$ (each factor is at m level, $q = m^{r-2}$)

Let $P = [F_1 \mid G_1, \ldots, G_q \mid G_{q+1}, \ldots, G_{2q}]$

Now one can obtain a UOSP d generated by the row space of the matrix P for an $(m^p) \times m^{2q}$ (Here $p = r-2$, $q = m^{r-2}$) experiment involving m^r ($r \geq 4$) runs for estimation of the mean, $(2q+1)$ M.E. $F_1, G_1, G_2, \ldots, G_{2q}$ and q 2FI $G_j G_{q+i}$ for any one $j = 1, 2, \ldots, q$ and for all $i = 1, 2, \ldots, q$ where $d \equiv (F_1 : G_1, G_2, \ldots, G_q : G_{q+1}, G_{q+2}, \ldots, G_{2q})_4$.

Example 3.3. Consider PG(3, 2) over GF(2). Here r = 4. Let H_1 and H_2 be any the two hyperplanes.

H_1 : (0011, 0101, 0110, 1000, 1011, 1101, 1110)

H_2 : (0011, 0101, 0110, 1001, 1010, 1100, 1111)

H_1 and H_2 intersect in an (r – 3) flat H_3

H_3 : (0011, 0101, 0110), $|H_3| = 3$

$|H_1 \cup H_2|$: (0011, 0101, 0110, 1000, 1011, 1101, 1110, 1001, 1010, 1100, 1111)

$|H_1 \cup H_2| = f = 11$. We divide these 11 points into three sets.

Set 1 : H_3 : (0011, 0101, 0110)

Set 2 : $H_1 \setminus H_3$: (1000, 1011, 1101, 1110)

Set 3 : $H_2 \setminus H_3$: (1001, 1010, 1100, 1111)

Assign the points in the sets as in

Set 1 : F_1 (the factor F_1 is at $2^2 = 4$ level, $p = 4 - 2 = 2$)

Set 2 : G_1, G_2, G_3, G_4 (at m = 2 level each)

Set 3 : G_5, G_6, G_7, G_8 (at m = 2 level each)

Let $P = [F_1 \mid G_1, \ldots, G_4 \mid G_5, \ldots, G_8]$

$$= \begin{bmatrix} F_1 & G_1 & G_2 & G_3 & G_4 & G_5 & G_6 & G_7 & G_8 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The row space of matrix P generates an optimal plan $d \equiv (F_1; G_1, \ldots, G_4; G_5, \ldots, G_8)_4$ for a 4×2^8 experiments involving 2^4 runs for estimation of the mean, 9 M.E. F_1, G_i ($i = 1, 2, \ldots, 8$) and 4 2FI $G_j G_{4+i}$ for any one $j = 1, \ldots, 4$ and for all $i = 1, \ldots, 4$.

Table 3.4

S.No.	PG(r – 1, m)	Optimal plan	Experiment
1.	PG(3, 2)	$d \equiv (F_1 : G_1, \ldots, G_4 : G_5, \ldots, G_8)_4$	4×2^8
2.	PG(4, 2)	$d \equiv (F_1 : G_1, \ldots, G_8 : G_9, \ldots, G_{16})_4$	8×2^{16}
3.	PG(3, 3)	$d \equiv (F_1 : G_1, \ldots, G_9 : G_{10}, \ldots, G_{18})_4$	9×3^{18}
4.	PG(4, 3)	$d \equiv (F_1 : G_1, \ldots, G_{27} : G_{28}, \ldots, G_{54})_4$	27×3^{54}

Theorem 3.5. For any prime power m, one can construct a UOSP d for an $(m^2) \times m^{w+1}$ ($w \geq 1$) experiment involving mr runs $r \geq 5$ for estimation of the mean, (w + 2) M.E. $G_0, G_1, \ldots, G_w, F_1$ and (w + 1) 2FI $G_0F_1, G_0G_1, \ldots, G_0G_w$ where $d \equiv (G_0; F_1, G_1, G_2, \ldots, G_w)_2$.

Proof. Let us take c = 1 in Lemma 3.2. Let H be an (r – c – 1) = (r – 2)-flat (hyperplane) in PG(r – 1, m). Let F_1 be a line contained in H. Let $G_1, G_2, \ldots, G_w$ (where $w = \dfrac{m^{r-1} - m^2}{m-1}$) be the points on H disjoint from F_1 : G_0 is a point (c – 1 = 0-flat) not included in H. The factors $G_0, \ldots, G_w$ are at 2 level each. The factor F_1 is at m^2 level. Let P denote a matrix with columns given by the unions of the points in $F_1, G_0, G_1, \ldots, G_w$.

$P = [G_0 \mid F_1, G_1, G_2, \ldots, G_w]$

Now one can obtain a UOSP d generated by the row space of the matrix P for an $(m^2) \times m^{w+1}$ experiment involving m^r runs for estimation of the mean, (w + 2) M.E. $G_0, G_1, G_2, ..., G_w, F_1$ and (w + 1) 2FI $G_0F_1, G_0G_1, ..., G_0G_w$ where $d \equiv (G_0; F_1, G_1, G_2, ..., G_w)_2$.

Example 3.4. Consider PG(4, 2) over GF(2). There are $\frac{2^4-1}{2-1} = 15$ points lying on any hyperplane (i.e. 3-flat). Let the points of a hyperplane H be (00010, 00100, 00110, 01000, 01010, 01100, 01110, 10000, 10010, 10100, 10110, 11000, 11010, 11100, 11110). Let F_1 be a line contained in H. The points lying on F_1 are (00010,00100,00110). $G_1, ... G_{12}$ are the points in $H \setminus F_1$. (11111) (say G_0) is a point in PG(4, 2) not lying in H. $G_0, G_1, ..., G_{12}$ are at 2 level each and F_1 is at 2^2 level. Let

$$P = [G_0 \mid F_1 \mid G_1 \mid ...G_{12}]$$

$$= \left[\begin{array}{c|cccccccccccc} G_0 & F_1 & G_1 & G_2 & G_3 & G_4 & G_5 & G_6 & G_8 & G_9 & G_{10} & G_{11} & G_{12} \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Then one can obtain an optimal plan $d \equiv (G_0; F_1, G_1, ..., G_{12})_2$ generated by the row space of the matrix P for a 4×2^{13} experiment involving 2^5 runs for estimation of the mean, 14 M.E. $F_1, G_0, G_1, ..., G_{12}$ and 13 2FI $G_0F_1, G_0G_1, ..., G_0G_{12}$.

Table 3.5

Sl.No.	PG(r – 1,m)	Optimal plan	Experiment
1.	PG(4, 2)	$d \equiv (G_0; F_1, G_1, ..., G_{12})_2$	4×2^{13}
2.	PG(5, 2)	$d \equiv (G_0; F_1, G_1, ..., G_{28})_2$	4×2^{29}
3.	PG(4, 3)	$d \equiv (G_0; F_1, G_1, ..., G_{36})_2$	9×3^{37}

Theorem 3.6. For any prime power m, one can construct a UOSP d for an $(m^2) \times m^w$ $(w \geq 1)$ experiment involving m^r runs $(r \geq 5)$ for estimation of the mean, (w + 1) M.E. $F_0, G_1, ..., G_w$ and w 2FI $F_0G_1, F_0G_2, ..., F_0G_w$ where $d \equiv (F_0; G_1, G_2, ..., G_w)_2$.

Proof. Let us take c = 2 in Lemma 3.2.

We can obtain a UOSP.

Table 3.6

Sl.No.	PG(r – 1, m)	Optimal plan	Experiment
1.	PG(4,2)	$d \equiv (F_0; G_1, ..., G_7)_2$	4×2^7
2.	PG(5,2)	$d \equiv (F_0; G_1, ..., G_{15})_2$	4×2^{15}
3.	PG(4,3)	$d \equiv (F_0; G_1, ..., G_{13})_2$	9×3^{13}

Theorem 3.7. For any prime power m, one can construct a UOSP d for an $(m^3) \times m^w$ $(w \geq 1)$ experiment involving m^r runs $r \geq 6$ for estimation of the mean, (w + 1) M.E. $F_0, G_1, ..., G_w$ and w 2FI $F_0G_1, F_0G_2, ..., F_0G_w$ where $d \equiv (F_0; G_1, G_2, ..., G_w)_2$.

Proof. Let us take c = 3 in Lemma 3.2.

We can obtain a UOSP.

Example 3.5. Consider PG(5, 2) over GF(2). Let us consider the planes F_0 and K such that they are disjoint. Let points of F_0 be (000001, 000010, 000011, 000100, 000101, 000110, 000111) and points of K be (001000, 010000, 011000, 100000, 101000, 110000, 111000).

Let us denote the points of K as $G_1, ..., G_7$.

Let $P = [F_0 \mid G_1 \mid ...G_7]$

Now, we can obtain an optimal plan $d \equiv (F_0; G_1, G_2, ...G_7)_2$ generated by the row space of the matrix P for a 8×2^7 experiment involving 2^6 runs for estimation of the mean, 8 M.E. $F_0, G_1, G_2, ...G_7$ and 7 2FI $F_0G_1, ..., F_0G_7$.

Table 3.7

Sl.No.	PG(r – 1,m)	Optimal plan	Experiment
1.	PG(5, 2)	$d \equiv (F_0; G_1, G_2, ...G_7)_2$	8×2^7
2.	PG(6, 2)	$d \equiv (F_0; G_1, G_2, ...G_{15})_2$	8×2^{15}
3.	PG(5, 3)	$d \equiv (F_0; G_1, G_2, ...G_{13})_2$	27×3^{13}

Theorem 3.8. For any prime power m, one can construct a UOSP d for an $(m^3)^2 \times m^w$ $(w \geq 1)$ experiment involving m^r, $r \geq 7$ runs for estimation of the mean, (w + 2) M.E. $F_0, F_1, G_1, ..., G_w$ and (w + 1) 2FI $F_0F_1, F_0G_1, ..., F_0G_w$ where $d \equiv (F_0; F_1, G_1, G_2, ..., G_w)_2$.

Proof. Let us take c = 3 in Lemma 3.2.

We can obtain a UOSP.

Table 3.8

Sl.No.	PG(r – 1, m)	Optimal plan	Experiment
1.	PG(6, 2)	$d \equiv (F_0; F_1, G_1, G_2, \ldots G_8)_2$	$8^2 \times 2^8$
2.	PG(7, 2)	$d \equiv (F_0; F_1, G_1, G_2, \ldots G_{24})_2$	$8^2 \times 2^{24}$
3.	PG(6, 3)	$d \equiv (F_0; F_1, G_1, G_2, \ldots G_{27})_2$	$(27)^2 \times 3^{27}$

Theorem 3.9. For any prime power m, one can construct a UOSP d for an $(m^p) \times (m^q)^w$, $p \geq 1$, $q = 2, 3$, $w \geq 5$ experiment involving m^r runs, $r \geq 5$ for estimation of the mean, (w + 1) M.E. $F_0, F_1, \ldots, F_w$ and w 2FI $F_0F_1, F_0F_2, \ldots, F_0F_w$ where $d \equiv (F_0; F_1, \ldots, F_w)_2$.

Proof.

Case I. Let us take c = 4 in Lemma 3.2. Let F_0 and K be an (r – 5)-flat and a 3 flat in PG(r – 1,m) respectively. The number of points lying on any line in PG(r – 1, m) is m + 1. The $\frac{m^4 - 1}{m - 1}$ points lying on the 3-flat K can be divided into $(m^2 + 1)$ (= w say) disjoint lines.

Let us denote these w lines as $F_1, F_2, \ldots, F_w$.

Here factor F_0 is at m^{r-4} level since F_0 is an (r – 5)-flat. $F_1, \ldots, F_w$ are at m^2 level each as they are disjoint lines in PG(r – 1,m).

Let $P = [F_0 \mid F_1, \ldots, F_w]$

Now one can obtain a UOSP d generated by the row space of the matrix P for an $(m^p) \times (m^2)^w$ (Here $p = (r - 4)$, $q = 2$, $w = (m^2 + 1)$) experiment involving m^r ($r \geq 5$) runs for estimation of the mean, (w + 1) M.E. F_0, $F_1, \ldots, F_w$ and w 2FI $F_0F_1, F_0F_2, \ldots, F_0F_w$ where

$$d \equiv (F_0; F_1, \ldots, F_w)_2.$$

Example 3.6. Consider PG(4, 2) over GF(2).

Here r = 5, c = 4, m = 2. F_0 is 0-flat and K is a 3-flat.

Let F_0 : (11111)

The 15 points of K can be divided into 5 disjoint lines, say (10000, 01100, 11100), (01000, 00110, 01110), (00100, 11010, 11110), (00010, 10100, 10110), (11000, 01010, 10010).

Let us denote these lines as $F_1, \ldots, F_5$.

$$P = [F_0 \mid F_1, \ldots, F_5]$$

Here the factor F_i (i = 1, ... , 5) is at 4 level where factor F_0 is at 2 level because F_i (i = 1, ... , 5) is a line and F_0 is a point.

P generates an optimal plan $d \equiv (F_0; F_1, \ldots, F_5)_2$ for 2×4^5 experiment involving 2^5 runs for estimation of the mean, 6 M.E. F_i (i = 0, 1, ... , 5) and 5 2FI F_0F_i (i = 1, ... , 5).

Table 3.9

Sl.No.	PG(r – 1, m)	Optimal plan	Experiment
1.	PG(4, 2)	$d \equiv (F_0; F_1, \ldots, F_5)_2$	2×4^5
2.	PG(6, 2)	$d \equiv (F_0; F_1, \ldots, F_5)_2$	8×4^5
3.	PG(4, 3)	$d \equiv (F_0; F_1, \ldots, F_{10})_2$	3×9^{10}
4.	PG(6, 3)	$d \equiv (F_0; F_1, \ldots, F_{10})_2$	27×9^{10}

Proof.

Case II. Let us take c = 6 in Lemma 3.2. Let F_0 and K be an (r – 7)-flat and a 5-flat respectively in PG(r – 1, m).

We can obtain a UOSP d.

Table 3.10

Sl.No.	PG(r – 1,m)	Optimal plan	Experiment
1.	PG(6, 2)	$d \equiv (F_0; F_1, \ldots, F_9)_2$	2×8^9
2.	PG(7, 2)	$d \equiv (F_0; F_1, \ldots, F_9)_2$	4×8^9
3.	PG(6, 3)	$d \equiv (F_0; F_1, \ldots, F_{28})_2$	3×27^{28}

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