Available online at www.isas.org.in/jisas
JOURNAL OF THE INDIAN SOCIETY OF
AGRICULTURAL STATISTICS 68(1) 2014 25-38

Balanced and Unbalanced Response Surface Designs Involving Qualitative Factors under Split-plot

Manohar Aggarwal[1], Sanjoy Roy Chowdhury[2], Anita Bansal[3] and Archana Verma[4]
[1]*Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA*
[2]*Lady Shri Ram College, University of Delhi, Delhi, India*
[3]*Ramjas College, University of Delhi, Delhi, India*
[4]*P.G.D.A.V. College, University of Delhi, Delhi, India*

Received 11 August 2011; Revised 18 September 2013; Accepted 07 October 2013

SUMMARY

Parker *et al*. (2006, 2007a, 2007b) have constructed balanced and unbalanced split-plot Box-Behnken designs and Central Composite designs involving quantitative factors using second-order split-plot response surface designs. In this paper we have developed balanced and unbalanced response surface designs under split-plot structure involving both quantitative factors and qualitative factors based on the designs given by Parker *et al*. (2006, 2007a, 2007b). In all these designs we have considered qualitative factor as a hard to change factor.

Keywords: Qualitative factors, Quantitative factors, Split-plot designs, Balanced designs, Unbalanced designs, Response surface designs, D-Optimal value.

1. INTRODUCTION

Box and Wilson (1951) introduced the concept of Response Surface Methodology (RSM) which has been discussed by many others. Some of the recent reviews are Neff and Myers (2000), Myers and Montgomery (2002), Myers *et al*. (2004), Box and Draper (2007), Andersoncook *et al*. (2009), Khuri and Mukhopadhyay (2010). Response Surface designs have been found to be efficient, economical and are useful for developing, improving and optimizing processes. The classical response surface designs have been developed using completely randomized designs assuming that the levels of all factors are equally easy to change. But in industries and manufacturing processes, often we come across situations involving two types of factors - hard to change (HTC) factors, whose levels are difficult to change, and easy to change (ETC) factors, whose levels are easy to change. In such situations complete randomization of the design is not appropriate, so split-plot design structure is used. Under split-plot nomenclature there are two distinct types of experimental units, each requiring separate randomization. HTC factors are randomly assigned to the whole-plot experimental units and ETC factors are assigned randomly to sub-plot units. This random assignment of factors gives rise to two error terms, whole-plots error δ and sub-plots error ε. When the numbers of sub-plot runs in the whole-plots are same, the design is said to be balanced and when the whole-plots are of different sizes then the design is said to be unbalanced. Kowalski (2002), McLeod and Brewster (2004), Vining *et al*. (2005), Montgomery (2005), Kowalski *et al*. (2007), Parker *et al*. (2008), Cheng and Tsai (2009), Wang *et al*. (2009) and Jones and Nachtsheim (2009) have discussed the concept of split-plot designs involving quantitative factors only.

Corresponding author: Sanjoy Roy Chowdhury
E-mail address : sanjoy.lsrc@gmail.com

Consider an agricultural experiment in response surface design under split-plot structure. We study the effects of irrigation methods (factor 1), types of pesticides (factor 2), fertilizers (factor 3) and qualities of seeds (factor 4) on the yield of a crop. In this experiment, fertilizers and seeds are ETC factors and irrigation methods and pesticides types are HTC factors. Irrigation methods and pesticides types require relatively larger areas as compared to fertilizers and seeds. All the four factors are quantitative in nature and are at three levels. HTC factors are randomly applied to whole-plot units. Within each whole-plot, ETC factors are randomly applied to sub-plot units.

The most popular second-order designs in response surface methodology are Central Composite Designs (CCDs) and Box-Behnken Designs (BBDs). Parker *et al*. (2006, 2007a, 2007b) have discussed two systematic methods for constructing balanced and unbalanced CCDs and BBDs under split-plot structure involving quantitative HTC and ETC factors. The first method was named as VKM method which is the generalized version of the method by Vining *et al*. (2005). The second method which minimizes the number of whole-plots was named as MWP method.

The above experiment may be studied using response surface designs under split-plot structure, given in Parker *et al*. (2006, 2007a, 2007b) for two HTC and two ETC quantitative factors. Here, we have two HTC and two ETC factors, all at three levels. They are quantitative in nature. Three-level balanced or unbalanced BBDs or CCDs with axial points for both HTC and ETC factors as ± 1 can be used. Irrigation methods and pesticides types are allocated to wholeplot units and fertilizers and seeds quality are allocated to sub-plot units. With two HTC and two ETC factors, balanced VKM method CCD and BBD require 10 whole-plots of size 4 each and unbalanced VKM method CCD and BBD require 9 whole-plots of size 4 each and 1 whole-plot of size 2. Balanced MWP method BBD needs 9 whole-plots of size 5 each.

Often in industrial experimentations, the experimenters need to conduct experiments when at least one of the factors is qualitative in nature. The response surface designs involving both qualitative and quantitative factors were studied and analyzed by Draper and John (1988), Aggarwal and Bansal (1998), Aggarwal *et al.* (2000), Ankenman and Dean (2003), Joseph *et al.* (2009) but none of them have constructed designs under split-plot structure.

In this paper we have developed balanced second-order response surface designs involving both qualitative and quantitative factors under split-plot structure using the balanced designs given by Parker *et al*. (2006, 2007a). We have also developed balanced and unbalanced response surface designs under split-plot structure involving both qualitative and quantitative factors using unbalanced designs given by Parker *et al*. (2007b). In all these designs we have considered only one hard to change qualitative factor, *s* hard to change quantitative factors and *r* easy to change quantitative factors. The selections of designs are made on the basis of D-optimal value.

2. MODEL AND DESIGN SELECTION CRITERION

Consider $(r + s + 1)$ number of factors, where r is the number of ETC quantitative factors, $x_1, x_2, \ldots, x_r$; s is the number of HTC quantitative factors, $z_1, z_2, \ldots, z_s$; and one HTC qualitative factor w. The second-order response surface model for the u^{th} run; $u = 1, 2, \ldots, N$, is given by the equation (2.1).

$$\begin{aligned} E(y_u) &= b_0 + t_0 w_u + \sum_{i=1}^{r} \beta_i x_{iu} + \sum_{i=1}^{s} \alpha_i z_{iu} + \sum_{i=1}^{r} \beta_{ii} x_{iu}^2 \\ &+ \sum_{i=1}^{s} \alpha_{ii} z_{iu}^2 + \sum_{i<j=1}^{r} \sum \beta_{ij} x_{iu} x_{ju} + \sum_{i=1}^{r} \sum_{j=1}^{s} \lambda_{ij} x_{iu} z_{ju} \\ &+ \sum_{i=1}^{r} \tau_i x_{iu} w_u + \sum_{i=1}^{s} \eta_i z_{iu} w_u \end{aligned} \tag{2.1}$$

with $\quad \text{var}(y_u) = \sigma_\delta^2 + \sigma_\varepsilon^2$ for all u

$$\text{cov}(y_u, y_{u'}) = \begin{cases} \sigma_\delta^2, & \text{for } u \neq u' \\ 0, & \text{otherwise} \end{cases}$$

where, β_0 is fixed but unknown; τ_0 is the effect due to the HTC qualitative factor w; β's are regression coefficients of ETC factors x; α's are regression coefficients of HTC quantitative factors z; λ_{ij} is the interaction coefficient between i^{th} ETC quantitative factor and j^{th} HTC quantitative factor; τ_i is the interaction coefficient between the qualitative factor w and i^{th} ETC quantitative factor; η is the interaction coefficient between the HTC qualitative factor w and i^{th} HTC quantitative factor. δ is the whole-plots error with σ_δ^2 whole-plots error variance and ε is the sub-plots error with σ_ε^2 sub-plots error variance and $\gamma = \sigma_\delta^2 / \sigma_\varepsilon^2$ is the variance ratio.

D-optimality, which is one of the several optimality criteria, is used for selecting the design.

3. ALGORITHM FOR THE CONSTRUCTION OF THE BALANCED AND UNBALANCED DESIGNS

3.1 Balanced Designs

The procedure for constructing second-order response surface designs under split-plot structure based on balanced CCDs and BBDs is as follows:

First, we consider designs with r ETC factors and (s + 1) HTC factors, developed by Parker *et al*. (2006, 2007a) to construct designs with r ETC factors, s HTC quantitative factors and one HTC qualitative factor w. Next we pick the first set of r and (s + 1) combination of the design. The $(s + 1)^{th}$ HTC factor is considered as a qualitative factor w. We have added one centre run to the designs given by Parker *et al*. (2006, 2007a) if the number of runs in the designs are odd. The design is then sorted on the basis of qualitative factor w. The zeroes present in the column of the qualitative factor w are replaced by a combination from all possible 2^p combinations of equal number of +1's and –1's, where p is the number of zeroes in the column of w, excluding number of zeroes in the center runs. This design is again sorted first on the basis of qualitative factor w and then on the basis of the remaining HTC quantitative factors. The model matrix is generated according to equation (2.1) and its D-optimal value is obtained. This procedure is repeated for all possible 2^p combinations having equal number of +1's and –1's. The same procedure is repeated for all combinations of r and (s + 1) factors of the VKM method CCDs and BBDs. We have further extended this procedure for developing designs for MWP method CCDs and BBDs. In all the CCDs we have assumed the value of α and β as 1. The above procedure is explained with the help of following example:

Example 1: Consider a MWP method BBD with parameter r = 3 (ETC quantitative factors), s = 1 (HTC quantitative factors) constructed by Parker *et al*. (2006). The MWP BBD is

MWP BBD

Runs	x_1	x_2	x_3	z_1	Runs	x_1	x_2	x_3	z_1	Runs	x_1	x_2	x_3	z_1
1	0	–1	–1	–1	14	0	–1	–1	1	27	0	–1	–1	0
2	0	1	–1	–1	15	0	1	–1	1	28	0	1	–1	0
3	0	–1	1	–1	16	0	–1	1	1	29	0	–1	1	0
4	0	1	1	–1	17	0	1	1	1	30	0	1	1	0
5	–1	0	–1	–1	18	–1	0	–1	1	31	–1	0	–1	0
6	1	0	–1	–1	19	1	0	–1	1	32	1	0	–1	0
7	–1	0	1	–1	20	–1	0	1	1	33	–1	0	1	0
8	1	0	1	–1	21	1	0	1	1	34	1	0	1	0
9	–1	–1	0	–1	22	–1	–1	0	1	35	–1	–1	0	0
10	1	–1	0	–1	23	1	–1	0	1	36	1	–1	0	0
11	–1	1	0	–1	24	–1	1	0	1	37	–1	1	0	0
12	1	1	0	–1	25	1	1	0	1	38	1	1	0	0
13	0	0	0	–1	26	0	0	0	1	39	0	0	0	0

There are N = 39 runs in the design, including one center run. The factor z_1 is considered as qualitative factor w. After sorting the design with respect to this factor w, the zeroes of the column are replaced with [–1 –1 –1 1 –1 1 1 1 –1 –1 1 1]', which is one of the combination from all 2^{12} possible combinations of ±1. In order to make the design balanced we have added another center run against quantitative factors and then to the above combination of ±1 for qualitative factor we have added –1 and +1 against two center runs of the quantitative factors. Next generate model matrix **X** as per equation (2.1).

The design is then sorted on the basis of HTC qualitative factor w in lexicographic order. The sorted model matrix **X** is as follows.

Model Matrix **X**

Runs	k	x_1	x_2	x_3	w	x_1x_1	x_2x_2	x_3x_3	x_2x_1	x_3x_1	x_3x_2	wx_1	wx_2	wx_3
1	1	–1	–1	0	1	1	1	0	–1	0	0	1	–1	0
2	1	–1	1	0	1	1	1	0	–1	0	0	–1	1	0
3	1	1	–1	0	1	1	1	0	1	0	0	–1	–1	0
4	1	1	1	0	1	1	1	0	1	0	0	1	1	0
5	1	–1	0	–1	1	1	0	1	–1	1	–1	0	0	0
6	1	–1	0	1	1	1	0	1	–1	–1	1	0	0	0
7	1	1	0	–1	1	1	0	1	1	–1	–1	0	0	0
8	1	1	0	1	1	1	0	1	1	1	1	0	0	0
9	1	0	–1	–1	1	0	1	1	0	0	–1	0	–1	–1
10	1	0	–1	1	1	0	1	1	0	0	1	0	–1	1
11	1	0	1	–1	1	0	1	1	0	0	–1	0	1	–1
12	1	0	1	1	1	0	1	1	0	0	1	0	1	1
13	1	0	0	0	1	0	0	0	0	0	0	0	0	0
14	1	1	1	0	1	1	1	0	1	0	0	1	1	0
15	1	–1	0	1	1	1	0	1	–1	–1	1	0	0	0
16	1	1	0	–1	1	1	0	1	1	–1	–1	0	0	0
17	1	1	0	1	1	1	0	1	1	1	1	0	0	0
18	1	0	1	–1	1	0	1	1	0	0	–1	0	1	–1
19	1	0	1	1	1	0	1	1	0	0	1	0	1	1
20	1	0	0	0	1	0	0	0	0	0	0	0	0	0
21	1	–1	–1	0	–1	1	1	0	–1	0	0	1	1	0
22	1	–1	1	0	–1	1	1	0	–1	0	0	–1	–1	0
23	1	1	–1	0	–1	1	1	0	1	0	0	–1	1	0
24	1	1	1	0	–1	1	1	0	1	0	0	1	–1	0
25	1	–1	0	–1	–1	1	0	1	–1	1	1	0	0	0
26	1	–1	0	1	–1	1	0	1	–1	–1	–1	0	0	0
27	1	1	0	–1	–1	1	0	1	1	–1	1	0	0	0
28	1	1	0	1	–1	1	0	1	1	1	–1	0	0	0
29	1	0	–1	–1	–1	0	1	1	0	0	1	0	1	–1
30	1	0	–1	1	–1	0	1	1	0	0	–1	0	1	1
31	1	0	1	–1	–1	0	1	1	0	0	1	0	–1	–1
32	1	0	1	1	–1	0	1	1	0	0	–1	0	–1	1
33	1	0	0	0	–1	0	0	0	0	0	0	0	0	0
34	1	–1	–1	0	–1	1	1	0	–1	0	0	1	1	0
35	1	–1	1	0	–1	1	1	0	–1	0	0	–1	–1	0
36	1	1	–1	0	–1	1	1	0	1	0	0	–1	1	0
37	1	–1	0	–1	–1	1	0	1	–1	1	1	0	0	0
38	1	0	–1	–1	–1	0	1	1	0	0	1	0	1	–1
39	1	0	–1	1	–1	0	1	1	0	0	–1	0	1	1
40	1	0	0	0	–1	0	0	0	0	0	0	0	0	0

where, k is the constant term. D-optimal value of sorted design involving qualitative and quantitative factors under split-plot structure is obtained using the formula $\frac{|X'\Sigma^{-1}X|^{1/t}}{N}$, where Σ is the variance-covariance matrix, for different values of variance ratio γ as follows:

Variance Ratio γ	0	0.1	0.2	0.3	0.4	1.0
D-optimal value	.4407	.3738	.3475	.3312	.3195	.2831

whereas the D-optimal value of the design given by Parker *et al*. (2006) involving only quantitative factors is .3857.

The second-order designs under split-plot structure, based on balanced VKM method BBDs and MWP method BBDs, involving both qualitative and quantitative factors are given in Table 1 and Table 2 respectively of Appendix I. Table 3 and Table 4 of Appendix I contains designs based on balanced VKM method CCDs and MWP method CCDs respectively. We have given only those designs in Appendix I which have high D-optimal values. The table gives the level combinations of HTC qualitative factor w and D-optimal values of the designs, where, D_1 gives the D-optimal value of the original design and D_2 gives the D-optimal value of our design. Only some of the results are shown in the Appendix I. The complete catalogue of the results is available with the authors.

3.2. Unbalanced Designs

The procedure for constructing unbalanced second-order response surface designs under split-plot structure based on VKM and MWP method CCDs and BBDs is same as that followed for constructing balanced designs. We pick the first set of the design generated by Parker *et al.* (2007b), with r ETC and $(s + 1)$ HTC quantitative factors, in order to construct the design for r ETC and s HTC quantitative and one HTC qualitative factor. The $(s + 1)^{th}$ factor is considered as qualitative factor w. One centre run is added to the designs given by Parker *et al.* (2007b) if the number of runs in the design is odd. The design is sorted on the basis of factor w. The zeroes present in the column of the qualitative factor w are replaced by a combination from all possible 2^p combinations of +1's and –1's, where p is the number of zeroes in the column of w, excluding number of zeroes in the center runs. When there are equal number of ±1 in the combination then we get balanced design and when the number of ±1 in the combination are unequal then we get unbalanced design. Thus two types of designs are constructed using the unbalanced designs given by Parker *et al*. (2007b), one balanced designs and other unbalanced designs.

This procedure for constructing the designs is explained with the help of the following example.

Example 2: Consider the unbalanced VKM method CCD given by Parker *et al*. (2007b) with $r = 3$ (ETC quantitative factors) and $s = 1$ (HTC quantitative factors). Taking $r = 3$, $s = 0$ and $w = 1$ (HTC qualitative factor), the unbalanced second-order response surface design under split-plot structure is constructed. The design is sorted on the basis of HTC qualitative factor w. The sorted model matrix **X**, obtained using equation (2.1), is as follows:

Model Matrix **X**

k	x_3	x_2	x_1	w	x_1x_1	x_2x_2	x_3x_3	x_2x_1	x_3x_1	x_3x_2	wx_1	wx_2	wx_3
1	–1	–1	–1	1	1	1	1	–1	1	–1	1	–1	–1
1	1	–1	–1	1	1	1	1	1	–1	–1	–1	–1	–1
1	–1	1	–1	1	1	1	1	–1	1	–1	–1	1	–1
1	1	1	–1	1	1	1	1	1	–1	–1	1	1	–1
1	–1	–1	1	1	1	1	1	–1	–1	1	1	–1	1
1	1	–1	1	1	1	1	1	1	1	1	–1	–1	1
1	–1	1	1	1	1	1	1	–1	–1	1	–1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0	0	0	1	0	0	0	0	0	0	0	0	0
1	–1	–1	–1	–1	1	1	1	–1	1	1	1	1	–1
1	1	–1	–1	–1	1	1	1	1	–1	1	–1	1	–1
1	–1	1	–1	–1	1	1	1	–1	1	1	–1	–1	–1
1	1	1	–1	–1	1	1	1	1	–1	1	1	–1	–1
1	–1	–1	1	–1	1	1	1	–1	–1	–1	1	1	1
1	1	–1	1	–1	1	1	1	1	1	–1	–1	1	1
1	–1	1	1	–1	1	1	1	–1	–1	–1	–1	–1	1
1	1	1	1	–1	1	1	1	1	1	–1	1	–1	1
1	0	0	–1	–1	0	0	1	0	0	1	0	0	0
1	0	0	1	–1	0	0	1	0	0	–1	0	0	0
1	0	–1	0	–1	0	1	0	0	0	0	0	1	0
1	0	1	0	–1	0	1	0	0	0	0	0	–1	0
1	–1	0	0	–1	1	0	0	–1	0	0	0	0	0
1	1	0	0	–1	1	0	0	1	0	0	0	0	0
1	0	0	0	–1	0	0	0	0	0	0	0	0	0

D-optimal value of sorted design involving qualitative and quantitative factors under split-plot structure is .5250 whereas the D-optimal value of the same design given by Parker *et al*. (2007b) involving only quantitative factors is .3920.

The balanced second-order response surface designs obtained using unbalanced VKM and MWP method BBDs given by Parker *et al*. (2007b), under split-plot structure involving both qualitative and quantitative factors are given in Table 5 and Table 6 of Appendix II respectively. Table 7 of Appendix II contains balanced second-order response surface designs under split-plot structure obtained using unbalanced VKM method CCDs given by Parker *et al*. (2007b). In Appendix III, Table 8 and Table 9 respectively gives unbalanced second-order response surface designs under split-plot structure obtained using unbalanced VKM method BBDs and MWP method BBDs given by Parker *et al*. (2007b) and Table 10 of Appendix III gives unbalanced second-order response surface designs under split-plot structure obtained using unbalanced VKM method CCDs given by Parker *et al*. (2007b). We have given only those designs in the Appendices II and III which have high D-optimal values. The tables give the level combinations of HTC qualitative factor *w,* D-optimal values of the designs given by Parker *et al*. (2007b), denoted by D_1 and D-optimal values of our designs, denoted by D_2. Only some of the results are shown in the Appendix II and Appendix III. The complete catalogue of the results is available with the authors.

4. CONCLUSIONS

It has been observed that when one of the HTC quantitative factors of the design given by Parker *et al*. (2006, 2007a, 2007b) is changed to HTC qualitative factor, the D-optimal value of most of the designs under split-plot structure increases as compared to original designs. The designs developed in this paper enable one to study both qualitative and quantitative factors under split-plot structure. It has also been seen that these designs give better estimates of quadratic effects and more effects are estimated independently.

ACKNOWLEDGEMENTS

The authors would like to thank the referees for constructive comments.

REFERENCES

Aggarwal, M.L. and Bansal, A. (1998). Robust response surface designs for quantitative and qualitative factors. *Commu. Statist.- Theory Methods*, **27**, 89-106.

Aggarwal, M.L., Gupta, B.C. and Bansal, A. (2000). Small robust response-surface designs for quantitative and qualitative factors. *Am. J. Math. Manage. Sci.*, **39**, 103-130.

Ankenman, B.C. and Dean, A.M. (2003). *Quality Improvement and Robustness via Design of Experiments*. Handbook of Statistics, Elsevier Science, vol **22**. Chapter 8, 263-317.

Anderson-Cook, C.M., Borror, C.M. and Montgomery, D.C. (2009). Response surface design evaluation and comparison. *J. Statist. Plann. Inf.*, **139**, 629-641.

Box, G.E.P. and Wilson, K.B. (1951). On the experimental attainment of optimum conditions. *J. Roy. Statist. Soc.,* **B13**, 1-45.

Box, G.E.P. and Draper, N.R. (2007). *Response Surfaces, Mixtures, and Ridge Analyses*. Wiley, New York.

Cheng, C.S. and Tsai, P.W. (2009). Optimal two level regular fractional factorial block and split-plot designs. *Biometrika,* **96**, 83-93.

Draper, N. and John, J.A. (1988). Response surface designs for quantitative and qualitative variables. *Technometrics,* **30**, 423-428.

Joseph, V.R., Ai, M. and Wu, C.F.J. (2009). Bayesian-inspired minimum aberration two – and – four – level designs. *Biometrika*, **96**, 95-106.

Jones, B. and Nachtsheim, C.J. (2009). Split-plot designs: What, why and how. *J. Qual Tech.*, **41**, 340-361.

Khuri, A.I. and Mukhopadhyay, S. (2010). Response surface methodology. In: *Wires Computational Statistics,* **2**, 128-149.

Kowalski, S.M. (2002). 24 Run split-plot experiments for robust parameter design. *J. Qual. Tech.*, **34**, 399-410.

Kowalski, S.M., Parker, P.A. and Vining, G.G. (2007). Tutorial: Industrial split-plot experiments. *Qual. Engg.*, **19**, 1-15.

McLeod, R.G. and Brewster, J.F. (2004). The design of blocked fractional factorial split-plot experiments. *Technometrics*, **46**, 135-146.

Montgomery, D.C. (2005). *Design and Analysis of Experiments*. 6th edition, John Wiley and Sons, New York.

Myers, R.H. and Montgomery, D.C. (2002). *Response Surface Methodology: Process and Product Optimization Using Designed Experiments*. 2nd edition, John Wiley and Sons, New York.

Myers, R.H., Montgomery, D.C., Vining, G.G., Borrow, C.M. and Kowalski, S.M. (2004). Response surface methodology: A retrospective and literature survey. *J. Qual. Tech.*, **36**, 53-77.

Neff, A.R. and Myers, R.H. (2000). *Recent Development in Response Surface Methodology and its Applications in Industry, Statistical Process Monitoring and Optimization*. Marcel Dekker, Inc. 457-481, Edited by Park, S.H. and Vining, G.G.

Parker, P.A., Anderson-Cook, C., Robinson, T.J. and Liang, L. (2008). Robust split-plot designs. *Qual. Reliability Engg. Intern.*, **24**, 107-121.

Parker, P.A., Kowalski, S.M. and Vining, G.G. (2006). Classes of split-plot response surface designs for equivalent estimation. *Qual. Reliability Engg. Intern.*, **22**, 291-305.

Parker, P.A., Kowalski, S.M. and Vining, G.G. (2007a). Construction of balanced equivalent estimation second-order split-plot designs. *Technometrics*, **49**, 56-64.

Parker, P.A., Kowalski, S.M. and Vining, G.G. (2007b). Unbalanced and minimal point equivalent estimation second-order split-plot designs. *J. Qual. Tech.*, **39**, 376-388.

Vining, G.G., Kowalski, S.M. and Montgomery, D.C. (2005). Response surface designs within a split-plot structure. *J. Qual. Tech.*, **37**, 115-129.

Wang, Li., Kowalski, S.M. and Vining, G.G. (2009). Orthogonal blocking of response surface split-plot designs. *J. Appl. Statist.*, **36**, 303-321.

APPENDIX I

Table 1. D-optimal values of the Balanced Designs based on Balanced VKM method BBDs.

r	*s*	*w*	*N*	D_1	D_2	Level Combinations of qualitative factor *w*														
2	0	1	16	.3536	.4008	−1	−1	1	1											
					.3977	−1	1	1	−1											
					.3912	−1	1	−1	1											
					.2850	1	1	−1	−1											
3	0	1	48	.3401	.4030	−1	−1	−1	−1	−1	−1	1	1	1	1	1	1			
					.4043	−1	−1	−1	−1	1	1	−1	1	1	1	−1	1			
					.4013	−1	−1	−1	−1	−1	1	1	1	1	1	1	−1			
					.3757	−1	−1	−1	−1	1	−1	1	−1	1	1	1	1			
4	0	1	96	.2367	.2690	−1	−1	−1	−1	1	1	1	1	1	−1	1	−1	1	1	1
						−1	1	−1	−1	−1	−1	1	−1	1						
					.2686	−1	−1	−1	−1	1	1	1	1	1	−1	1	−1	1	1	−1
						−1	1	1	−1	1	−1	1	−1	−1						
					.2681	−1	−1	−1	−1	1	1	1	1	1	−1	−1	−1	1	−1	1
						−1	1	1	1	1	−1	1	−1	−1						
					.2522	−1	−1	−1	−1	1	1	1	1	1	−1	−1	−1	1	1	1
						−1	1	1	−1	1	−1	−1	1	−1						
1	1	1	18	.2234	.3450	−1	1	1	−1											
					.3407	1	1	−1	−1											
					.3370	1	−1	1	−1											
					.3360	−1	−1	1	1											
2	1	1	40	.2246	.2753	1	1	1	1	−1	−1	−1	−1	1	−1	1	−1			
					.2746	1	−1	1	1	−1	−1	1	−1	−1	−1	1	1			
					.2691	−1	−1	−1	−1	−1	−1	1	1	1	1	1	1			
					.2512	1	−1	−1	−1	−1	1	−1	−1	1	1	1	1			
3	1	1	120	.3799	.4303	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	−1	1	−1
						1	1	1	1	1	1	1	1	−1	−1	−1	1	−1	1	1
						1	1	−1	1	1	1									
					.4314	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	−1	1	−1
						1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	1	1
						−1	1	1	1	1	1									
					.4318	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	−1	1	1
						−1	−1	−1	1	1	1	−1	1	1	1	1	1	1	1	1
						1	1	−1	1	1	−1									
					.4233	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	−1	1	−1
						1	1	1	1	1	1	1	1	−1	−1	−1	1	−1	1	1
						1	1	1	−1	1	1									
4	1	1	80	.2059	.2368	−1	−1	−1	−1	−1	1	1	−1	−1	−1	1	−1	1	1	1
						−1	1	−1	1	1	1	1	1	−1						
					.2267	−1	−1	−1	−1	−1	1	−1	1	1	1	1	−1	1	1	1
						1	1	−1	−1	1	1	−1	−1	−1						
					.2242	−1	−1	−1	−1	−1	1	−1	1	1	1	1	−1	1	1	1
						1	1	−1	−1	−1	1	1	−1	−1						
					.2092	−1	−1	−1	−1	−1	1	−1	1	1	1	1	−1	1	1	1
						1	1	−1	1	−1	−1	−1	−1	1						

Table 2. D-optimal values of the Balanced Designs based on Balanced MWP method BBDs.

r	*s*	*w*	*N*	D_1	D_2	Level Combinations of qualitative factor *w*											
2	0	1	16	.3575	.4140	1	–1	1	–1								
					.4008	–1	–1	1	1								
					.3937	–1	1	1	–1								
					.3867	1	1	–1	–1								
3	0	1	40	.3857	.4409	–1	–1	–1	–1	–1	–1	1	1	1	1	1	1
					.4333	–1	–1	–1	–1	–1	1	1	–1	1	1	1	1
					.4422	–1	–1	–1	–1	1	1	–1	1	1	1	–1	1
					.4407	–1	–1	–1	1	–1	1	1	1	–1	–1	1	1
4	0	1	28	.4269	.4596	–1	–1	–1	–1	1	1	1	1				
					.4306	–1	–1	–1	1	–1	1	1	1				
					.4290	–1	1	1	–1	–1	–1	1	1				
					.4284	–1	1	1	1	–1	–1	1	–1				

Table 3. D-optimal values of the Balanced Designs based on Balanced VKM method CCDs.

r	*s*	*w*	*N*	D_1	D_2	Level Combinations of qualitative factor *w*														
1	0	1	12	.4283	.5546	–1	1													
					.5546	1	–1													
2	0	1	24	.3527	.3995	–1	–1	1	1											
					.4005	–1	1	–1	1											
					.3004	–1	1	1	–1											
					.3982	1	–1	–1	1											
3	0	1	64	.2933	.3427	–1	–1	–1	1	1	–1	1	1	1	–1	–1	1	–1	1	1
						–1	–1	–1	–1	1	1	1	1	–1						
					.3424	–1	–1	–1	1	1	–1	1	1	1	1	1	–1	–1	–1	–1
						1	–1	1	–1	1	–1	1	–1	1						
					.3399	–1	–1	–1	1	1	–1	1	1	1	–1	–1	1	1	–1	1
						–1	1	–1	1	–1	1	–1	–1	1						
					.3035	–1	–1	–1	1	1	–1	1	1	1	–1	–1	1	–1	1	1
						–1	–1	–1	1	–1	–1	1	1	1						
4	0	1	48	.2801	.2951	–1	–1	–1	–1	1	1	1	1							
					.2959	–1	–1	–1	1	–1	1	1	1							
					.2957	1	–1	–1	1	1	–1	–1	1							
					.2947	1	1	–1	–1	–1	–1	1	1							
1	1	1	20	.3527	.4783	–1	–1	–1	1	1	1									
					.4789	–1	–1	1	1	–1	1									
					.4784	–1	1	–1	–1	1	1									
					.4511	–1	–1	1	1	1	–1									

2	1	1	40	.3538	.3990	–1 –1 –1 –1 –1 –1 1 1 1 1 1 1
					.4060	–1 –1 –1 1 –1 –1 –1 1 1 1 1 1
					.4024	–1 –1 –1 1 1 –1 1 –1 1 –1 1 1
					.3990	–1 –1 1 1 –1 1 1 1 1 –1 –1 –1
3	1	1	48	.317	.3547	1 1 –1 –1 1 –1 1 –1 1 1 1 –1 –1 –1 1
						–1 1 1 –1 –1
					.3548	1 1 1 –1 –1 1 1 1 –1 –1 1 –1 1 –1 1
						–1 1 –1 –1 –1
					.3069	1 1 –1 –1 1 1 –1 –1 –1 –1 1 1 –1 –1 1
						1 –1 –1 1 1
					.3518	1 1 –1 1 –1 1 –1 1 –1 –1 1 –1 1 1 1
						–1 –1 –1 –1 1
4	1	1	80	.3041	.3255	–1 –1 –1 1 –1 1 1 –1 –1 1 –1 1 –1 1 1
						1 –1 –1 1 –1 1 –1 1 1
					.3263	–1 –1 –1 1 –1 1 1 1 –1 –1 –1 1 –1 –1 –1
						1 –1 1 1 1 1 1 1 –1
					.3259	–1 –1 –1 1 –1 1 1 1 1 –1 –1 –1 –1 –1 1
						–1 –1 1 1 –1 1 1 1 1
					.3063	–1 –1 –1 1 –1 1 1 –1 –1 1 –1 1 1 –1 –1
						–1 –1 –1 1 1 1 1 1 1

Table 4. D-optimal values of the Balanced Designs based on Balanced MWP method CCDs.

r	*s*	*w*	*N*	D_1	D_2	Level Combinations of qualitative factor *w*
1	0	1	10	.4622	.5903	–1 1
					.5903	1 –1
2	0	1	16	.4472	.5083	–1 –1 1 1
					.4716	–1 1 –1 1
					.4452	–1 1 1 –1
					.3677	1 1 –1 –1
3	0	1	28	.4211	.4726	–1 –1 –1 1 1 1
					.4602	–1 –1 1 –1 1 1
					.4507	1 1 –1 –1 –1 1
					.4465	1 –1 –1 1 1 –1
4	0	1	28	.4269	.4596	–1 –1 –1 –1 1 1 1 1
					.4306	–1 –1 –1 1 –1 1 1 1
					.4284	–1 1 1 1 –1 –1 1 –1
					.4130	1 –1 –1 1 –1 1 1 –1

APPENDIX II

Table 5. D-optimal values of the Balanced Designs based on Unbalanced VKM method BBDs.

r	*s*	*w*	*N*	D_1	D_2	Level Combinations of qualitative factor *w*
2	0	1	14	.377	.4255	–1 –1 1 1
					.4133	–1 1 –1 1
					.4009	–1 1 1 –1
3	0	1	38	.3812	.4412	–1 –1 –1 –1 1 1 –1 1 1 1 –1 1
					.4399	–1 –1 –1 –1 –1 –1 1 1 1 1 1 1
					.4397	–1 1 1 1 –1 –1 1 –1 1 1 –1 –1
					.4306	–1 –1 –1 –1 –1 1 1 –1 1 1 1 1
4	0	1	74	.2728	.3041	–1 –1 –1 –1 1 1 1 1 1 –1 1 –1 1 1 1
						–1 1 –1 –1 –1 –1 1 –1 1
					.3037	–1 –1 –1 –1 1 1 1 1 1 –1 1 –1 1 1 1
						–1 –1 1 –1 –1 1 –1 1 –1
					.3031	–1 –1 –1 –1 1 1 1 1 1 –1 –1 –1 1 –1 1
						–1 1 1 1 1 –1 1 –1 –1
					.3005	–1 –1 –1 –1 1 1 1 1 1 –1 –1 1 –1 1 1
						–1 1 1 1 1 –1 –1 –1 –1
2	1	1	38	.3805	.2857	1 –1 –1 1 1 1 –1 –1 1 –1 –1 1
					.2804	1 –1 1 1 1 –1 –1 –1 1 1 –1 –1
					.2755	–1 –1 –1 –1 –1 –1 1 1 1 1 1 1
					.2625	–1 –1 –1 –1 –1 1 –1 1 1 1 1 1
3	1	1	110	.3805	.4279	–1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 1 –1 –1 1
						1 1 –1 1 –1 –1 1 –1 1 1 1 1 1 1 1
						1 -1 1 1 1 1
					.4233	–1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 1 –1 –1 1
						1 1 1 1 1 1 1 1 1 1 1 1 –1 –1 1
						1 –1 1 –1 –1 1
					.4161	–1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 1 –1 –1 1
						1 1 –1 1 –1 –1 1 –1 1 1 1 1 1 1 1
						1 1 –1 1 1 1
					.4063	–1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 1 –1 –1 1
						1 1 1 –1 –1 1 –1 –1 1 1 1 1 1 1 1
						1 1 1 1 1 –1
4	1	1	74	.2118	.2467	–1 –1 –1 1 –1 1 1 1 –1 1 1 1 1 1 –1
						1 –1 1 1 –1 –1 –1 –1 –1
					.2444	–1 –1 –1 1 –1 1 1 –1 1 –1 –1 1 1 1 1
						–1 1 –1 –1 1 1 1 –1 –1
					.2332	–1 –1 –1 1 –1 1 1 –1 –1 1 –1 1 –1 1 1
						1 –1 –1 1 –1 1 –1 1 1
					.2288	–1 –1 –1 1 1 –1 –1 –1 1 –1 –1 –1 –1 –1 –1
						1 1 1 1 1 1 1 1 1

Table 6. D-optimal values of the Balanced Designs based on Unbalanced MWP method BBDs.

r	*s*	*w*	*N*	D_1	D_2	Level Combinations of qualitative factor *w*
2	0	1	14	.3788	.4377	1 –1 1 –1
					.4255	–1 –1 1 1
					.4133	–1 1 –1 1
3	0	1	26	.2531	.3389	1 1 1 –1 1 1 –1 –1 –1 –1 1 –1
					.3304	1 –1 1 –1 1 1 –1 –1 –1 –1 1 1
					.3264	–1 –1 1 1 1 1 –1 1 1 –1 –1 –1
					.3219	–1 –1 –1 –1 –1 –1 1 1 1 1 1 1
4	0	1	42	.1729	.2254	–1 –1 –1 –1 –1 1 –1 1 1 1 1 –1 1 1 1
						1 1 –1 –1 –1 1 1 –1 –1
					.2209	–1 –1 –1 –1 1 –1 –1 –1 1 1 1 –1 1 1 1
						1 –1 –1 –1 –1 1 1 1 1
					.2194	–1 –1 –1 –1 1 –1 –1 –1 1 1 –1 1 1 1 1
						1 1 1 1 1 –1 –1 –1 –1
					.2143	–1 –1 –1 –1 –1 1 1 –1 1 1 1 1 1 1 1
						1 –1 1 1 –1 –1 –1 –1 –1

Table 7. D-optimal values of the Balanced Designs based on Unbalanced VKM method CCDs.

r	*s*	*w*	*N*	D_1	D_2	Level Combinations of qualitative factor *w*
2	0	1	22	.3724	.4243	–1 1 –1 1
					.4227	–1 –1 1 1
					.3095	–1 1 1 –1
3	0	1	24	.3920	.5199	–1 –1 –1 1 1 1
					.5077	1 –1 1 –1 –1 1
					.5045	–1 1 1 –1 –1 1
4	0	1	42	.3127	.3318	–1 –1 –1 1 –1 1 1 1
					.3305	–1 –1 –1 –1 1 1 1 1
					.3302	1 –1 1 –1 1 –1 –1 1
2	1	1	38	.3674	.4189	–1 1 –1 1 –1 1 1 –1 1 1 –1 –1
					.4188	–1 –1 –1 1 1 1 1 –1 –1 1 1 –1
					.4139	–1 –1 –1 –1 –1 –1 1 1 1 1 1 1
					.4098	–1 –1 1 –1 –1 1 1 –1 –1 1 1 1
3	1	1	32	.3375	.4367	–1 –1 –1 –1 –1 –1 –1 1 1 1 1 1 1 1
					.3865	–1 –1 1 1 1 1 –1 1 –1 –1 –1 1 1 –1
					.3828	–1 –1 1 1 1 1 1 1 –1 –1 –1 1 –1 –1
					.3112	–1 –1 1 1 –1 –1 1 1 –1 –1 –1 1 1 1
4	1	1	74	.3244	.3483	–1 –1 –1 1 1 –1 –1 1 –1 –1 –1 1 1 –1 1
						–1 1 1 1 1 1 –1 –1 1
					.3434	–1 –1 –1 1 1 –1 –1 1 1 1 –1 1 1 –1 1
						1 –1 –1 1 –1 1 –1 1 –1
					.3429	–1 –1 –1 1 1 –1 –1 1 1 –1 –1 1 1 1 1
						–1 1 –1 –1 1 1 –1 –1 1
					.3344	–1 –1 –1 1 1 –1 –1 1 –1 –1 –1 1 1 –1 1
						–1 1 1 1 1 1 1 –1 –1

APPENDIX III

Table 8. D-optimal values of the Unbalanced Designs based on Unbalanced VKM method BBDs.

r	*s*	*w*	*N*	D_1	D_2	Level Combinations of qualitative factor *w*
2	0	1	14	.3770	.4509	–1 –1 –1 –1
					.4229	–1 1 1 1
					.4199	–1 –1 1 –1
					.3874	–1 1 –1 –1
3	0	1	38	.3812	.4405	–1 –1 1 –1 1 –1 1 1 –1 –1 1 –1
					.4396	–1 –1 1 –1 1 –1 1 –1 1 –1 1 –1
					.4388	–1 –1 –1 –1 –1 –1 –1 1 1 –1 1 1
					.4337	–1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1
4	0	1	74	.2728	.3039	–1 –1 –1 –1 –1 –1 1 1 1 1 –1 1 –1 1 1
						1 –1 1 –1 1 –1 –1 1 –1
					.3033	–1 –1 –1 –1 –1 –1 1 1 1 1 –1 1 –1 1 1
						–1 1 –1 –1 1 1 –1 1 –1
					.3031	–1 –1 –1 –1 –1 –1 1 1 –1 1 1 –1 –1 1 1
						1 –1 1 –1 1 1 1 1 1
					.3009	–1 –1 –1 –1 –1 –1 1 1 –1 1 1 –1 –1 1 –1
						–1 –1 1 1 –1 1 1 –1 –1
2	1	1	38	.3805	.2840	1 1 –1 –1 1 1 –1 1 –1 1 –1 1
					.2836	1 1 –1 –1 1 1 –1 –1 1 1 1 –1
					.2822	1 1 –1 –1 1 1 1 –1 –1 1 –1 1
					.2725	–1 –1 –1 –1 –1 –1 –1 –1 1 1 1 1
3	1	1	110	.3805	.4222	–1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1
						–1 –1 –1 1 –1 1 –1 1 1 1 1 1 1 –1 –1
						1 –1 –1 –1 –1 1
					.4203	–1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1
						–1 –1 –1 1 –1 1 –1 1 1 1 1 1 1 –1 –1
						1 –1 –1 –1 –1 –1
					.4197	–1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1
						–1 –1 –1 1 –1 1 1 –1 –1 –1 –1 –1 –1 –1 –1
						–1 –1 –1 –1 –1 1
					.4189	–1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1 –1
						–1 –1 –1 1 –1 1 1 –1 –1 –1 –1 –1 1 –1 –1
						–1 –1 –1 –1 –1 –1
4	1	1	74	.2118	.2472	–1 –1 –1 –1 –1 –1 1 –1 –1 –1 –1 –1 1 1 1
						1 –1 –1 –1 –1 –1 –1 –1 –1
					.2433	–1 –1 –1 –1 –1 –1 –1 1 1 1 1 –1 1 1 –1
						1 1 1 1 1 –1 1 1 1
					.2423	–1 –1 –1 –1 –1 –1 1 –1 –1 –1 –1 –1 –1 1 1
						1 1 1 –1 –1 –1 –1 –1 –1
					.241	–1 –1 –1 –1 –1 –1 –1 1 1 1 1 1 1 1 –1
						–1 –1 1 1 –1 –1 1 1 –1

Table 9. D-optimal values of the Unbalanced Designs based on Unbalanced MWP method BBDs.

r	*s*	*w*	*N*	D_1	D_2	Level Combinations of qualitative factor *w*
2	0	1	14	.3788	.4509	–1 –1 –1 –1
					.4199	–1 –1 1 –1
					.3986	–1 1 1 1
3	0	1	26	.2531	.3358	–1 –1 1 1 1 –1 1 –1 –1 –1 –1 –1
					.3292	–1 –1 –1 –1 –1 –1 –1 –1 1 1 1 –1
					.3214	–1 –1 –1 –1 –1 –1 –1 1 1 1 1 –1
					.3138	–1 –1 –1 –1 –1 –1 –1 –1 –1 1 –1 1
4	0	1	42	.1729	.2331	–1 –1 –1 –1 –1 –1 –1 –1 1 1 1 –1 1 1 1
						1 –1 1 1 –1 –1 –1 –1 –1
					.2318	–1 –1 –1 –1 –1 –1 –1 –1 1 1 1 –1 1 1 1
						1 –1 1 1 1 –1 –1 –1 –1
					.2298	–1 –1 –1 –1 –1 –1 –1 –1 1 1 1 –1 1 1 1
						1 –1 1 1 1 –1 –1 1 –1

Table 10. D-optimal values of the Unbalanced Designs based on Unbalanced VKM method CCDs.

r	*s*	*w*	*N*	D_1	D_2	Level Combinations of qualitative factor *w*
2	0	1	22	.3724	.4306	–1 –1 –1 –1
					.4245	–1 –1 –1 1
					.4217	–1 1 1 1
3	0	1	24	.392	.5246	–1 –1 –1 –1 –1 –1
					.5160	–1 –1 –1 –1 –1 1
					.5097	–1 –1 –1 –1 1 –1
4	0	1	42	.3127	.3333	–1 –1 –1 –1 –1 –1 –1 –1
					.3319	–1 –1 –1 –1 –1 1 1 1
					.3318	–1 –1 –1 –1 1 –1 –1 1
					.3302	–1 –1 –1 1 –1 1 1 –1
2	1	1	38	.3674	.4166	1 –1 1 1 –1 –1 –1 –1 –1 1 1 –1
					.4155	–1 –1 –1 –1 1 –1 1 –1 1 –1 1 –1
					.4144	–1 –1 –1 –1 –1 –1 –1 1 1 1 1 1
					.3968	–1 –1 1 1 –1 –1 1 1 –1 –1 –1 –1
3	1	1	32	.3375	.4105	1 1 –1 1 1 –1 –1 –1 –1 –1 1 1 1 1
					.4025	1 1 1 1 –1 1 1 –1 –1 1 –1 –1 1 –1
					.3999	1 –1 –1 1 1 1 1 –1 1 1 1 1 –1 –1
					.3978	–1 1 –1 1 1 1 –1 1 1 1 –1 –1 1 1
4	1	1	74	.3244	.3416	–1 –1 –1 –1 –1 –1 1 –1 1 1 1 –1 –1 –1
						1 1 –1 1 1 –1 –1 1 1 1
					.3316	–1 –1 –1 –1 –1 –1 1 –1 1 1 1 –1 –1 –1
						1 1 –1 1 1 1 –1 –1 1 –1
					.3313	–1 –1 –1 –1 –1 –1 1 –1 1 1 1 –1 –1 –1
						1 1 1 1 1 1 –1 1 –1 –1
					.3293	–1 –1 –1 –1 –1 –1 1 –1 1 1 1 –1 –1 –1
						1 1 –1 1 1 1 –1 1 1 –1