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On a Bivariate Version of Alternative Hyper-Poisson Distribution

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SUMMARY

Here we introduce a bivariate version of the alternative hyper-Poisson distribution (*AHPD*) of Kumar and Nair (Statistica, 2012) as the distribution of the random sum of bivariate Bernoulli random variables. It is shown that both marginal distributions of this bivariate version are *AHPD*s and derive some important aspects of the distribution such as expressions for its probability mass function, factorial moments and conditional distributions. Certain recurrence relations for its probabilities, raw moments and factorial moments are also obtained. Further, the maximum likelihood estimation of the parameters of the distribution is discussed and illustrated using a real life data set.

Keywords: Confluent hypergeometric function, Displaced Poisson distribution, Factorial moment generating function, Hermite distribution, Poisson distribution, Probability generating function.

1. INTRODUCTION

Bardwell and Crow (1964) considered a class of two parameter discrete distribution, namely the hyper-Poisson distribution (*HPD*) through the following probability mass function (*p.m.f.*), for $x = 0, 1, ...$

$$f(x) = \frac{\Gamma(\gamma)}{\Gamma(\gamma+x)} \frac{\theta^x}{\phi(1;\gamma;\theta)}, \tag{1}$$

in which γ, θ are positive reals and $\phi(1; \gamma; \theta)$ is the confluent hypergeometric series (for details see Mathai and Houbold 2008 or Slater 1960). When $\gamma = 1$, the *HPD* reduces to Poisson distribution and when γ is a positive integer, the distribution is known as the displaced Poisson distribution studied by Staff (1964). Bardwell and Crow (1964) termed the distribution as sub-Poisson when $\gamma < 1$ and super-Poisson when $\gamma > 1$. Various methods of estimation of the parameters of the *HPD* have been discussed in Bardwell and Crow (1964), Crow and Bardwell (1965) and Roohi and Ahmad (2003a). Some queuing theory associated with hyper-Poisson arrivals has been considered in Nisida (1962). Roohi and Ahmad (2003b) obtained certain recurrence relations for negative moments and ascending factorial moments of the *HPD*. Kemp (2002) developed *q*-analogue of the *HPD* and Ahmad (2007) introduced and studied Conway-Maxwell hyper-Poisson distribution. Kumar and Nair (2011, 2012) introduced certain extended versions of the *HPD* and discussed some of its applications. Ahmad (1981) considered a bivariate version of the *HPD* (which we named as the bivariate hyper-Poisson distribution or in short the *BHPD*) through the probability generating function (*p.g.f.*)

$$Q(t_1, t_2) = [\phi_1(\eta_1)\phi_2(\eta_2)]^{-1} \exp[\eta(t_1-1)(t_2-1)] \, \phi_1(\eta_1 t_1)\, \phi_2(\eta_2 t_2), \tag{2}$$

in which $\phi_i(x) = \phi(1; \lambda_i; x)$. For $r \geq 0$, $s \geq 0$, the *p.m.f.* $q(r, s) = P[Z_1 = r, Z_2 = s]$ of $Z = (Z_1, Z_2)$ following *BHPD* with *p.g.f.* (2) is

$$q(r,s) = \frac{e^{\eta}\Gamma(\lambda_1)\Gamma(\lambda_2)}{\phi_1(\eta_1)\phi_2(\eta_2)} \sum_{i=0}^{\min(r,s)} \sum_{j=0}^{r-i} \sum_{k=0}^{s-i} \frac{(-1)^{j+k}\eta_1^{r-i-j}\eta_2^{s-i}}{\Gamma(\lambda_1+r-i-j)} \frac{\eta^{i+j+k}}{\Gamma(\lambda_2+s-i-k)i!j!k!}, \tag{3}$$

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where $\lambda_1 > 0$, $\lambda_2 > 0$ and $0 < \eta \le min\ [\eta_1/\lambda_1), (\eta_2/\lambda_2)]$.

Kumar and Nair (2012) considered an alternative form of the *HPD*, namely the Alternative hyper-Poisson distribution (*AHPD*), through the following *p.m.f.* for $x = 0, 1, \ldots$.

$$g(x) = \frac{\theta^x}{(\gamma)_x}\phi(1+x;\gamma+x;-\theta) \qquad (4)$$

in which $\gamma > 0$, $\theta > 0$ and $(a)_x$ is the rising factorial:

$$(a)_k = a(a+1)\ldots(a+k-1) = \Gamma(a+k)/\Gamma(a),$$

for $k = 1, 2, \ldots$ and $(a)_0 = 1$. The *p.g.f.* of the *AHPD* is

$$G(s) = \phi[1;\ \gamma;\ \theta(s-1)]. \qquad (5)$$

An interesting property of the *AHPD* is that it is under-dispersed when $\gamma < 1$ and over-dispersed when $\gamma > 1$.

Through the present paper we introduce a bivariate version of the *AHPD* and termed it as 'the bivariate alternative hyper-Poisson distribution (*BAHPD*)' and obtain some of its important properties. In section 2, it is shown that the *BAHPD* possess a random sum structure and both of its marginals are *AHPD*s. Further we obtain expressions for its conditional probability distribution, probability mass function and factorial moments in section 2. In section 3, we develop certain recurrence relations for probabilities, raw moments and factorial moments of the *BAHPD* and in section 4 we discuss the estimation of the parameters of the *BAHPD* by method of maximum likelihood. The *BAHPD* has been fitted to a well-known data set and it is observed that the *BAHPD* gives better fit than the bivariate Poisson distribution (*BPD*) and the *BHPD* of Ahmad (1981) with *p.g.f.* (2).

Note that both the *BPD* and the *BHPD* do not have a random sum structure, while the *BAHPD* possess a bivariate random sum structure as shown in section 2. The random sum structure arises in several areas of scientific research including actuarial science, agricultural sciences, biological sciences and physical sciences. Chapter 9 of Johnson *et al.* (2005) fully devoted to univariate random sum distributions.

2. THE BAHP DISTRIBUTION

Let Y be a non-negative integer valued random variable following the *AHPD* with *p.g.f.* (5) in which $\theta = \theta_1 + \theta_2 + \theta_3$, $\theta_1 > 0$, $\theta_2 > 0$ and $\theta_3 \ge 0$. Define $\beta_j = \frac{\theta_j}{\theta}$, for $j = 1, 2, 3$. Consider the sequence $\{X_n = (X_{1n}, X_{2n}), n \ge 1\}$ of independently and identically distributed bivariate Bernoulli random variables, each with *p.g.f.*

$$A(t_1, t_2) = \beta_1 t_1 + \beta_2 t_2 + \beta_3 t_1 t_2.$$

Assume that $Y, X_1, X_2, \ldots$ are independent. Let $S_0 = (S_{10}, S_{20}) = (0, 0)$ and define

$$S_Y = (S_{1Y}, S_{2Y}) = \left(\sum_{y=1}^{Y} X_{1y}, \sum_{y=1}^{Y} X_{2y}\right)$$

Then the *p.g.f* of S_Y is the following,

$$H(t_1, t_2) = G\{A(t_1, t_2)\}$$
$$= \phi[1;\ \gamma;\ \theta_1(t_1-1) + \theta_2(t_2-1) + \theta_3(t_1t_2-1)] \qquad (6)$$

A distribution with *p.g.f.* given in (6) we call 'the bivariate alternative hyper-Poisson distribution' or in short, the *BAHPD*. Clearly the *BAHPD* with $\gamma = 1$ is the bivariate Poisson distribution discussed in Kocherlakotta and Kocherlakotta (1992, pp.90) and when γ is a positive integer the *BAHPD* with *p.g.f.* (6) reduces to the *p.g.f.* of a bivariate version of an alternative form of the displaced Poisson distribution.

Let (Y_1, Y_2) be a random variable having the *BAHPD* with *p.g.f.* (6). Then the marginal *p.g.f.* of Y_1 and Y_2 are respectively

$$H_{Y_1}(t) = H(t, 1) = \phi[1;\ \gamma;\ (\theta_1 + \theta_3)(t-1)].$$

and

$$H_{Y_2}(t) = H(1, t) = \phi[1;\ \gamma;\ (\theta_2 + \theta_3)(t-1].$$

Let y be a non-negative integer such that $P(Y_1 = y) > 0$. On differentiating (6) with respect to t_2, y times and putting $t_1 = t$ and $t_2 = 0$, we get

$$H^{(0,y)}(t,0) = [\theta_2 + \theta_3 t]^y \left(\prod_{j=0}^{y-1} D_j\right)\delta_y(t,0) \qquad (7)$$

where $D_j = \frac{1+j}{\gamma+j}$ and $\delta_j(t_1, t_2) = \phi[1+j;\ \gamma+j;\ \theta_1(t_1-1) + \theta_2(t_2-1) + \theta_3(t_1t_2-1)]$ for $j = 0, 1, 2, \ldots$. Now applying the formula for the *p.g.f.* of the conditional distribution in terms of partial derivatives of the joint *p.g.f.*, developed by Subrahmaniam (1966), we obtain the conditional *p.g.f.* of Y_1 given $Y_2 = y$ as

$$H_{Y_1|Y_2=y}(t) = \left(\frac{\theta_2+\theta_3 t}{\theta_2+\theta_3}\right)^y \frac{\phi[1+y;\gamma+y;-(\theta_2+\theta_3)+\theta_1(t-1)]}{\phi[1+y;\gamma+y;-(\theta_2+\theta_3]} \quad (8)$$

$$= H_1(t)\, H_2(t) \quad (9)$$

where $H_1(t)$ is the *p.g.f.* of a binomial random variable with parameters y and $p = \theta_3(\theta_2 + \theta_3)^{-1}$ and $H_2(t)$ is the *p.g.f.* of a random variable following a generalized version *AHPD* with parameters $1+y$, $\gamma+y$, and θ_1. Note that, when $\theta_3 = 0$ and/or when $Y = 0$, $H_1(t)$ reduces to the *p.g.f.* of a random variable degenerate at zero. Thus the conditional distribution Y_1 given $Y_2 = y$ given in (8) can be viewed as the distribution of the sum of independent random variables V_1 with *p.g.f.* $H_1(t)$ and V_2 with *p.g.f.* $H_2(t)$. Consequently, from (9) we obtain the following.

$$E(Y_1|Y_2 = y) = \frac{y\theta_3}{(\theta_2+\theta_3)} + \frac{\theta_1 D_y \delta_{y+1}(1,0)}{\delta_y(1,0)} \quad (10)$$

$$Var(Y_1|Y_2 = y) = \frac{y\theta_2\theta_3}{(\theta_2+\theta_3)^2} + \frac{\theta_1 D_y}{\delta_y^2(1,0)}\{D_{y+1}\delta_y(1,0)\delta_{y+2}(1,0)\theta_1 + \delta_y(1,0)\delta_{y+1}(1,0) - D_y[\delta_{y+1}(1,0)]^2\theta_1\} \quad (11)$$

In a similar approach, for a non-negative integer y with $P(Y_1 = y) > 0$, we can obtain the conditional *p.g.f.* of Y_2 given $Y_1 = y$ by interchanging θ_1 and θ_2 in (8). Therefore it is evident that comments similar to those in case of the conditional distribution of Y_1 given $Y_2 = y$ are valid regarding conditional distribution of Y_2 given $Y_1 = y$ and explicit expressions for $E(Y_2|Y_1 = y)$ and $Var(Y_2|Y_1 = y)$ can be obtained by interchanging θ_1 and θ_2 in the right hand side expressions of (10) and (11) respectively.

On differentiating the *p.g.f.* $H(t_1, t_2)$ of *BAHPD* with respect to t_1 and t_2, r times and s times respectively, we get

$$H^{(r,s)}(t_1,t_2) = \left(\prod_{i=0}^{r-1} D_i\right)\sum_{m=0}^{\min(r,s)}\binom{s}{m}\frac{r!}{(r-m)!}\theta_3^m(\theta_1+\theta_3 t_2)^{r-m} \left(\prod_{i=r}^{r+s-m-1} D_i\right)(\theta_2+\theta_3 t_1)^{s-m}\delta_{r+s-m}(t_1,t_2) \quad (12)$$

Now, by putting $(t_1, t_2) = (0,0)$ in (12) and by dividing r! s!, we get the *p.m.f.* of the *BAHPD* as

$$h(r,s) = \theta_1^r\theta_2^s\sum_{m=0}^{\min(r,s)}\frac{D^*(m+1)\delta_{r+s-m}(0,0)}{m!(r-m)!(s-m)!}\left(\frac{\theta_3}{\theta_1\theta_2}\right)^m, \quad (13)$$

in which $D^*(m+1) = \prod_{j=0}^{r+s-m-1} D_j$. By putting $(t_1, t_2) = (1, 1)$ (in (12) we get the $(r, s)^{th}$ factorial moment $\mu_{[r, s]}$ of the *BAHPD* as

$$\mu_{[r,s]} = r!s!(\theta_1+\theta_3)^r(\theta_2+\theta_3)^s\sum_{m=0}^{\min(r,s)}\frac{D^*(m+1)}{m!(r-m)!(s-m)!}\alpha^m, \quad (14)$$

where $\alpha = \theta_3(\theta_1 + \theta_3)^{-1}(\theta_2 + \theta_3)^{-1}$. From (14) we obtain

$$E(Y_1) = \mu_{[1,0]} = \gamma^{-1}(\theta_1 + \theta_3),$$
$$E(Y_2) = \mu_{[0,1]} = \gamma^{-1}(\theta_2 + \theta_3),$$

and

$$Cov(Y_1, Y_2) = \mu_{[1,1]} - \mu_{[1,0]}\mu_{[0,1]} = \frac{(\gamma-1)}{\gamma^2(\gamma+1)}(\theta_1+\theta_3)(\theta_2+\theta_3)+\gamma^{-1}\theta_3,$$

Since $D_0 = \gamma^{-1}$ and $D_1 = 2(\gamma + 1)^{-1}$.

From the expression of Cov (Y_1, Y_2) it can be observed that Y_1 and Y_2 are partially correlated when $\gamma \geq 1$.

3. RECURRENCE RELATIONS

Let (Y_1, Y_2) be a random vector following the *BAHPD* with *p.g.f.* (6). For $j = 0, 1, 2 \ldots$, define $\gamma^* + j = (1 + j, \gamma + j)$ Now, the *p.m.f.* $h(r, s)$ of the *BAHPD* given in (13) we denote $h(r, s; \gamma^*)$. Then we have the following result in the light of relations:

$$H(t_1, t_2) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} h(r,s;\gamma^*)t_1^r t_2^s = \phi[1;\gamma;\theta_1(t_1-1)+\theta_2(t_2-1)+\theta_3(t_1t_2-1)] \quad (15)$$

and

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} h(r,s;\gamma^*+1)t_1^r t_2^s = \phi[2;\gamma+1;\theta_1(t_1-1) + \theta_2(t_2-1) + \theta_3(t_1t_2-1)] \quad (16)$$

Now we obtain certain recurrence relations for probabilities of the *BAHPD* through the following result.

Result 3.1. The probability mass function $h(r, s; \gamma^*)$ of the *BAHPD* distribution satisfies the following recurrence relations:

$$h(r+1, 0; \gamma^*) = \frac{\gamma^{-1}\theta_1}{(r+1)} h(r, 0; \gamma^* + 1), r \geq 0 \quad (17)$$

$$h(r+1, s; \gamma^*) = \frac{\gamma^{-1}}{(r+1)}[\theta_1 h(r, s; \gamma^* + 1) + \theta_3 h(r, s-1; \gamma^* + 1)], \; r \geq 0, s \geq 1 \quad (18)$$

$$h(0, s+1; \gamma^*) = \frac{\gamma^{-1}\theta_2}{(s+1)} h(0, s; \gamma^* + 1), s \geq 0 \quad (19)$$

$$h(r, s+1; \gamma^*) = \frac{\gamma^{-1}}{(s+1)}[\theta_2 h(r, s; \gamma^* + 1) + \theta_3 h(r-1, s; \gamma^* + 1)], \; r \geq 1, s \geq 0 \quad (20)$$

Proof. On differentiating (15) with respect to t_1, we obtain

$$H^{(1,0)}(t_1, t_2) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(r+1)h(r+1, s; \gamma^*)t_1^r t_2^s$$
$$= D_0(\theta_1 + \theta_3 t_2)\delta_1(t_1, t_2), \quad (21)$$

where $\delta_1(t_1, t_2)$ is as given in (7).

By applying (16) in (21) we get the following

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(r+1)h(r+1, s; \gamma^*)t_1^r t_2^s = D_0\left[\theta_1 \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}h(r, s; \gamma^* + 1)t_1^r t_2^s + \theta_3 \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}h(r, s; \gamma^* + 1)t_1^r t_2^{s+1}\right] \quad (22)$$

On equating the coefficient of $t_1^r t_2^0$ on both sides of (22) we get the relation (17) and on equating the coefficient of $t_1^r t_2^s$ on both sides of (22) we get the relation (18). We omit the proof of relations (19) and (20) as it is similar to that of relations (17) and (18).

Result 3.2. For r, s ≥ 0 simple recurrence relations for factorial moments $\mu_{[r, s]}(\gamma^*)$ of order (r, s) of the *BAHPD* are the following.

$$\mu_{[r+1, s]}(\gamma^*) = D_0 (\theta_1 + \theta_3) \mu_{[r, s]} (\gamma^* + 1) + D_0\theta_3 s\mu_{[r, s-1]} (\gamma^* + 1) \quad (23)$$

$$\mu_{[r, s+1]}(\gamma^*) = D_0(\theta_2 + \theta_3)\mu_{[r, s]}(\gamma^* + 1) + D_0\theta_3 r\mu_{[r-1, s]}(\gamma^* + 1) \quad (24)$$

in which $\mu_{[0, 0]}(\gamma^*) = 1$.

Proof. Let (Y_1, Y_2) be a random vector having the *BAHPD* with *p.g.f.* $H(t_1, t_2)$ as given in (6). Then the factorial moment generating function $F(t_1, t_2)$ of the *BAHPD* is

$$F(t_1, t_2) = H(1 + t_1, 1 + t_2)$$
$$= \phi[1; \gamma; (\theta_1 + \theta_3)t_1 + (\theta_2 + \theta_3)t_2 + \theta_3 t_1 t_2]$$
$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\mu_{[r, s]}(\gamma^*)\frac{t_1^r t_2^s}{r!s!} \quad (25)$$

On differentiating (25) with respect to t_1 and adopting a similar argument as in the proof of Result 3.1, one can obtain the following.

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\mu_{[r+1,s]}(\gamma^*)\frac{t_1^r t_2^s}{r!s!} = D_0\left\{(\theta_1 + \theta_3)\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\mu_{[r,s]}(\gamma^* + 1)\frac{t_1^r t_2^s}{r!s!} + \theta_3 \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\mu_{[r,s]}(\gamma^* + 1)\frac{t_1^r t_2^{s+1}}{r!s!}\right\} \quad (26)$$

Now on equating the coefficients of $(r!s!)^{-1} t_1^r t_2^s$ on both sides of (26) we obtain the relation (23). A similar procedure implies (24).

In a similar way as in the proof of Result 3.2, we can obtain certain recurrence relations for raw moments of the *BAHPD* from its characteristic function instead of considering the factorial moment generating function. Thus we have the following result.

Result 3.3. Two recurrence relations for the $(r, s)^{th}$ raw moments $\mu_{r, s}(\gamma^*)$ of the *BAHPD* are:

$$\mu_{r+1,s}(\gamma^*) = D_0\theta_1 \sum_{j=0}^{r}\binom{r}{j}\mu_{r-j,s}(\gamma^* + 1) + D_0\theta_3 \sum_{j=0}^{r}\sum_{k=0}^{s}\binom{r}{j}\binom{s}{k}\mu_{r-j, s-k}(\gamma^* + 1) \quad (27)$$

and

$$\mu_{r,s+1}(\gamma^*) = D_0\theta_2 \sum_{k=0}^{s}\binom{s}{k}\mu_{r,s-k}(\gamma^* + 1) + D_0\theta_3 \sum_{j=0}^{r}\sum_{k=0}^{s}\binom{r}{j}\binom{s}{k}\mu_{r-j,s-k}(\gamma^* + 1). \quad (28)$$

4. ESTIMATION OF PARAMETERS

Here we discuss the estimation of the parameters of the *BAPHD* by method of maximum likelihood.

Let $B(r, s)$ be the observed frequency of the $(r, s)^{th}$ cell of a bivariate data. Let y be the highest value of r observed and z be the highest value of s observed. Then by using (13) the likelihood function of the sample is the following.

$$L = \prod_{r=0}^{y}\prod_{s=0}^{z}[h(r,s)]^{B(r,s)},$$

which implies

$$\log L = \sum_{r=0}^{y}\sum_{s=0}^{z}B(r,s)\log h(r,s). \quad (29)$$

Let $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3$ and $\hat{\gamma}$ denote the maximum likelihood estimators of the respective parameters $\theta_1, \theta_2, \theta_3$ and γ of the *BAHPD*. On differentiating (29) partially with respect to the parameters and equates to zero, we have the following likelihood equations, in which

$$\Delta(r,s) = \sum_{m=0}^{\min(r,s)} \frac{D^*(m+1)\theta_1^{r-m}\theta_2^{s-m}\theta_3^{m}\delta_{r+s-m}(0,0)}{m!(r-m)!(s-m)!},$$

$\eta(a;\, x) = \psi(a) - \psi(a + x)$;

$\psi(a) = \frac{1}{\Gamma(a)}\frac{d}{da}[\Gamma(a)]$ and $\delta_j(0\,,\,0)$ is as defined in (7).

$$\frac{\partial \log L}{\partial \theta_1} = 0$$

equivalently,

$$\sum_{r=0}^{y}\sum_{s=0}^{z}B(r,s)\frac{1}{h(r,s)}$$

$$\times\left[\sum_{m=0}^{\min(r,s)} \frac{D^*(m+1)\theta_1^{r-m-1}\theta_2^{s-m}\theta_3^{m}\delta_{r+s-m}(0,0)}{(r-m-1)!(s-m)!m!} - \Delta(r,s)\right] = 0 \quad (30)$$

$$\frac{\partial \log L}{\partial \theta_2} = 0$$

equivalently,

$$\sum_{r=0}^{y}\sum_{s=0}^{z}B(r,s)\frac{1}{h(r,s)}$$

$$\times\left[\sum_{m=0}^{\min(r,s)} \frac{D^*(m+1)\theta_1^{r-m}\theta_2^{s-m-1}\theta_3^{m}\delta_{r+s-m}(0,0)}{(r-m)!(s-m-1)!m!} - \Delta(r,s)\right] = 0 \quad (31)$$

$$\frac{\partial \log L}{\partial \theta_3} = 0$$

equivalently,

$$\sum_{r=0}^{y}\sum_{s=0}^{z}B(r,s)\frac{1}{h(r,s)}$$

$$\times\left[\sum_{m=0}^{\min(r,s)} \frac{D^*(m+1)\theta_1^{r-m}\theta_2^{s-m}\theta_3^{m-1}\delta_{r+s-m}(0,0)}{(r-m)!(s-m)!(m-1)!} - \Delta(r,s)\right] = 0 \quad (32)$$

and

$$\frac{\partial \log L}{\partial \gamma} = 0$$

equivalently

$$\sum_{r=0}^{y}\sum_{s=0}^{z}B(r,s)\frac{1}{h(r,s)}\sum_{m=0}^{\min(r,s)} \frac{D^*(m+1)\theta_1^{r-m}\theta_2^{s-m}\theta_3^{m}}{(r-m)!(s-m)!m!}$$

$$\times\left\{\sum_{x=0}^{\infty} \frac{(1+r+s-m)_x[-(\theta_1+\theta_2)]^x}{(\gamma+r+s-m)_x\, x!}\,\eta(\gamma+r+s-m;x)\right.$$

$$\left. +\delta_{r+s-m}(0,0)\eta(\gamma;r+s-m)\right\} = 0 \quad (33)$$

Now $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3$ and $\hat{\gamma}$ can be obtained by solving the likelihood equations (30), (31), (32) and (33), by using some mathematical softwares.

5. AN APPLICATION

The procedure of maximum likelihood estimation discussed in section 4 is illustrated here using a real life data set taken from Patrat (1993). The description of data is as follows: The North Atlantic coastal states in USA can be affected by tropical cyclones. They divided the states into three geographical zones:

Zone 1: Texas, Loinsiane, The Mississipi, Albama;

Zone 2: Florida;

Zone 3: Other states.

Now the interest is in the study of the joint distribution of the pair (Y_1, Y_2) where Y_1 and Y_2 are the yearly frequency of hurricanes affecting respectively zone 1 and zone 3. The observed values of (Y_1, Y_2) during 93 years from 1899 to 1991 are as given in

Table 1. Comparison of observed and theoretical frequencies Hurricanes (1899-1991) having affected Zone 1 and Zone 3, using method of maximum likelihood.

Zone 1 / Zone 3		0	1	2	3	Total
0	OBS	27	9	3	2	41
	BPD	29.33	13.20	2.97	0.45	45.92
	BHPD	27.76	13.95	2.61	0.30	44.62
	BAHPD	23.81	15.36	2.28	0.19	41.64
1	OBS	24	13	1	0	38
	BPD	20.03	9.63	2.31	0.37	32.34
	BHPD	19.56	10.41	2.13	0.27	32.37
	BAHPD	25.98	10.79	1.91	0.19	38.87
2	OBS	8	2	1	0	11
	BPD	6.84	3.50	0.89	0.15	11.38
	BHPD	7.16	4.01	0.89	0.12	12.18
	BAHPD	6.52	3.23	0.69	0.09	10.53
3	OBS	1	0	2	0	3
	BPD	1.56	0.85	0.23	0.09	2.73
	BHPD	1.77	1.04	0.25	0.04	3.10
	BAHPD	0.95	0.57	0.15	0.02	1.69
Total	OBS	60	24	7	2	93
	BPD	59.52	26.06	5.66	0.82	93
	BHPD	56.25	29.41	5.88	0.73	92
	BAHPD	57.26	29.95	5.03	0.49	93

Table 2. Estimated values of the parameters of the *BPD*, the *BHPD* and the *BAHPD* by the method of maximum likelihood estimation and corresponding chi-square values.

Distributions	Estimates of parameters	Chi-square values
BPD	$\hat{\theta}_1 = 0.683$	1.967
	$\hat{\theta}_2 = 0.450$	
	$\hat{\theta}_3 = 0.021$	
BHPD	$\hat{\theta}_1 = 0.780$	1.865
	$\hat{\theta}_2 = 0.324$	
	$\hat{\theta} = 0.021$	
	$\hat{\lambda}_1 = 1.075$	
	$\hat{\lambda}_2 = 0.619$	
BAHPD	$\hat{\theta}_1 = 0.362$	1.707
	$\hat{\theta}_2 = 0.214$	
	$\hat{\theta}_3 = 0.043$	
	$\hat{\gamma} = 0.571$	

Table 1. We obtain the corresponding expected frequencies by fitting bivariate Poisson distribution (*BPD*), the bivariate hyper-Poisson distribution (*BHPD*) of Ahmad (1981) and the bivariate alternative hyper-Poisson distribution (*BAHPD*) to the data using method of maximum likelihood. The estimated values of the parameters of the *BPD*, the *BHPD* and the *BAHPD*, and the corresponding X^2- values are listed in Table 2. From Table 2 it can be observed that the *BAHPD* gives a better fit to this data compared to the existing model *BPD* and *BHPD*.

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